GLOBAL PROPERTIES OF BICONSERVATIVE SURFACES IN $\mathbb{R}^3$ AND $\mathbb{S}^3$

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Abstract. In this paper we survey some recent results on biconservative surfaces in 3 - dimensional space forms with a special emphasis for the case $c = 0$ and $c = 1$. We study the local and global properties of such surfaces, from extrinsic and intrinsic point of view. We obtain complete surfaces in $\mathbb{R}^3$ and $\mathbb{S}^3$.

1. Introduction

In the last decade, from the theory of *biharmonic submanifolds*, arised the study of *biconservative submanifolds* that imposed itself as a very promising and interesting research topic through papers like [3, 4, 5, 11, 23, 24].

Let us consider the *bienergy functional* defined on all smooth maps between two Riemannian manifolds (M^m, g) and (N^n, h) and given by

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \, v_g, \qquad \varphi \in C^\infty(M, N),$$

where $\tau(\varphi)$ is the tension field of φ. A critical point of E_2 is called a *biharmonic map*, and it is characterized by the vanishing of the *bitension field* $\tau_2(\varphi)$ (see[16]).

A Riemannian immersion $\varphi : M^m \to (N^n, h)$ or, simply, a submanifold M of N, is called *biharmonic* if φ is a biharmonic map.

Now, if $\varphi : M \to (N, h)$ is a fixed map, then E_2 can be thought as a functional on the set of all Riemannian metrics on M. This new functional's critical points are Riemannian metrics determined by the vanishing of the *stress-bienergy tensor* S_2. This tensor field satisfies

$$\operatorname{div} S_2 = -\langle \tau_2(\varphi), d\varphi \rangle.$$

If $\operatorname{div} S_2 = 0$ for a submanifold M in N, then M is called a biconservative submanifold and it is characterized by the fact that the tangent part of its bitension field vanishes.

The paper is organized in five sections as follows. After a section where we recall some notions and general results about biconservative submanifolds, we present in *Section 3* the local, intrinsic characterisation of biconservative surfaces. While by "local" we mean the biconservative surfaces $\varphi : M^2 \to N^3(c)$ with $f > 0$ and $\operatorname{grad} f \neq 0$ at any point of M, by "global" we mean the *complete* biconservative surfaces $\varphi : M^2 \to N^3(c)$ with $f > 0$ at any point of M and $\operatorname{grad} f \neq 0$ at any point of an open and dense subset of M. More precisely, the intrinsic characterisation theorem provides the necessary and sufficient conditions for an abstract surface $\left(M^2, g\right)$ to admit, locally, a biconservative embedding with $f > 0$ and $\operatorname{grad} f \neq 0$ at any point of M.

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Our main goal is to extend the *local* classification results for biconservative surfaces in $N^3(c)$, with $c = 0$ and $c = 1$, to *global* results, i.e., we ask that biconservative surfaces to be *complete* and with $|\operatorname{grad} f| > 0$ on an open dense subset.

In *Section 4* we consider the global problem and construct complete biconservative surfaces in $\mathbb{R}^3$ with $f > 0$ on M and $\operatorname{grad} f \neq 0$ at any point of an open dense subset of M. We determine the simply connected complete Riemannian surfaces $\left(\mathbb{R}^2, g_C\right)$ which admit a biconservative immersion in $\mathbb{R}^3$ in two ways: on one hand we use the local explicit parametric equation of biconservative surfaces in $\mathbb{R}^3$, and then we glue these local surfaces at the level of C^∞ smoothness, and one the other hand we use the intrinsic characterisation from previous section. Moreover, these immersions are explicitly given and they have $|\operatorname{grad} f| > 0$ on an open dense subset of $\mathbb{R}^2$.

In the *last section* we consider the global problem of biconservative surfaces in $\mathbb{S}^3$ with $f > 0$ on M and $\operatorname{grad} f \neq 0$ at any point of an open dense subset of M. As in $\mathbb{R}^3$ case, we use the local extrinsic classification of biconservative surfaces in $\mathbb{S}^3$, but now the "gluing" process is not as clear as in $\mathbb{R}^3$. Further, we change the point of view and use the intrinsic characterization of biconservative surfaces in $\mathbb{S}^3$. We determine the simply connected complete Riemannian surfaces $\left(\mathbb{R}^2, g_{C_1,C_1^*}\right)$ which admit a biconservative immersion in $\mathbb{S}^3$ and we show that, up to an isometry of $\mathbb{S}^3$, there exists only a one-parameter family of such Riemannian surfaces indexed by C_1.

2. Biconservative submanifolds; general properties

Throughout this work, all manifolds, metrics, maps are assumed to be smooth, i.e. in the C^∞ category, and we will often indicate the various Riemannian metrics by the same symbol $\langle , \rangle$. All surfaces are assumed to be connected and oriented.

A *harmonic map* $\varphi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds is a critical point of the *energy functional*

$$E : C^\infty(M, N) \to \mathbb{R}, \quad E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 \ v_g,$$

and it is characterized by the vanishing of its *tension field*

$$\tau(\varphi) = \operatorname{trace}_g \nabla d\varphi.$$

The idea of the stress-energy tensor associated to a functional comes from D. Hilbert ([15]). Given a functional E one can associate to it a symmetric 2-covariant tensor field S such that $\operatorname{div} S = 0$ at the critical points of E. When E is the energy functional, P. Baird and J. Eells ([1]), and A. Sanini ([25]), defined the tensor field

$$S = e(\varphi)g - \varphi^* h = \frac{1}{2}|d\varphi|^2 g - \varphi^* h,$$

and proved that

$$\operatorname{div} S = -\langle \tau(\varphi), d\varphi \rangle.$$

Thus, S can be chosen as the stress-energy tensor of the energy functional. It is worth mentioning that S has a variational meaning. Indeed, we can fix a map $\varphi : M^m \to (N^n, h)$ and think E as being defined on the set of all Riemannian metrics on M. The critical points of this new functional are Riemannian metrics determined by the vanishing of their stress-energy tensor S.

More precisely, we assume that M is compact and denote

$$\mathcal{G} = \{g \ : \ g \text{ is a Riemannian metric on } M\}.$$

For a deformation $\{g_t\}$ of g we consider $\omega = \frac{d}{dt}\big|_{t=0} g_t \in T_g\mathcal{G} = C\left(\odot^2 T^*M\right)$. We define the new functional

$$\mathcal{F} : \mathcal{G} \to \mathbb{R}, \quad \mathcal{F}(g) = E(\varphi)$$

and we have the following result.

Theorem 2.1 ([1, 25])**.** *Let $\varphi : M^m \to (N^n, h)$ and assume that M is compact. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}\left(g_t\right) = \frac{1}{2}\int_M \langle \omega, e(\varphi)g - \varphi^* h\rangle \ v_g.$$

Therefore g is a critical point of $\mathcal{F}$ if and only if its stress-energy tensor S vanishes.

We mention here that, if $\varphi : M^m \to N^n$ is an arbitrary isometric immersion, then $\operatorname{div} S = 0$.

A natural generalization of harmonic maps is given by biharmonic maps. A *biharmonic map* $\varphi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds is a critical point of the *bienergy functional*

$$E_2 : C^\infty(M, N) \to \mathbb{R}, \quad E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \ v_g,$$

and it is characterized by the vanishing of its *bitension field*

$$\tau_2(\varphi) = -\Delta^\varphi \tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi,$$

where

$$\Delta^\varphi = -\operatorname{trace}_g\left(\nabla^\varphi\nabla^\varphi - \nabla^\varphi_\nabla\right)$$

is the rough Laplacian of $\varphi^{-1}TN$ and the curvature tensor field is

$$R^N(X,Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z, \quad \forall X, Y, Z \in C(TM).$$

We remark that the *biharmonic equation* $\tau_2(\varphi) = 0$ is a fourth-order non-linear elliptic equation and that any harmonic map is biharmonic. A non-harmonic biharmonic map is called proper biharmonic.

In [17], G. Y. Jiang defined the stress-energy tensor S_2 of the bienergy (also called *stress-bienergy tensor*) by

$$\begin{aligned} S_2(X,Y) =& \frac{1}{2}|\tau(\varphi)|^2\langle X, Y\rangle + \langle d\varphi, \nabla\tau(\varphi)\rangle\langle X, Y\rangle \\ & - \langle d\varphi(X), \nabla_Y\tau(\varphi)\rangle - \langle d\varphi(Y), \nabla_X\tau(\varphi)\rangle, \end{aligned}$$

as it satisfies

$$\operatorname{div} S_2 = -\langle \tau_2(\varphi), d\varphi\rangle.$$

The tensor field S_2 has a variational meaning, as in the harmonic case. We fix a map $\varphi : M^m \to (N^n, h)$ and define a new functional

$$\mathcal{F}_2 : \mathcal{G} \to \mathbb{R}, \quad \mathcal{F}_2(g) = E_2(\varphi).$$

Then we have the following result.

Theorem 2.2 ([18])**.** *Let $\varphi : M^m \to (N^n, h)$ and assume that M is compact. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}_2\left(g_t\right) = -\frac{1}{2}\int_M \langle \omega, S_2\rangle \ v_g,$$

so g is a critical point of $\mathcal{F}_2$ is and only if $S_2 = 0$.

We mention that, if $\varphi : M^m \to N^n$ is an isometric immersion then $\operatorname{div} S_2$ does not necessarily vanish.

A submanifold of a given Riemannian manifold (N^n, h) is a pair (M^m, φ), where M^m is a manifold and $\varphi : M \to N$ is an immersion. We always consider on M the induced metric $g = \varphi^* h$, thus $\varphi : (M, g) \to (N, h)$ is an isometric immersion; for simplicity we will write $\varphi : M \to N$ without mentioning the metrics. Also, we will write $\varphi : M \to N$, or even M, instead of (M, φ).

A submanifold $\varphi : M^m \to N^n$ is called *biharmonic* if the isometric immersion φ is a biharmonic map from (M^m, g) to (N^n, h).

Even if the notion of biharmonicity may be more appropriate for maps than for submanifolds, as the domain and the codomain metrics are fixed and the variation is made only through the maps, the biharmonic submanifolds proved to be an interesting notion (see, for example [22])

In order to fix the notations, we recall here only the fundamental equations of first order of a submanifold in an Riemannian manifold. These equations define the second fundamental form, the shape operator and the connection in the normal bundle. Let $\varphi : M^m \to N^n$ be an isometric immersion. For each $p \in M$, $T_{\varphi(p)}N$ splits as an orthogonal direct sum

$$T_{\varphi(p)}N = d\varphi(T_pM) \oplus d\varphi(T_pM)^\perp, \tag{2.1}$$

and $NM = \bigcup_{p \in M} d\varphi(T_pM)^\perp$ is referred to as the normal bundle of φ, or of M, in N.

Denote by ∇ and ∇^N the Levi-Civita connections on M and N, respectively, and by ∇^φ the induced connection in the pull-back bundle $\varphi^{-1}(TN) = \bigcup_{p \in M} T_{\varphi(p)}N$. Taking into account the decomposition in (2.1), one has

$$\nabla^\varphi_X d\varphi(Y) = d\varphi(\nabla_X Y) + B(X, Y), \qquad \forall\, X, Y \in C(TM),$$

where $B \in C(\odot^2 T^*M \otimes NM)$ is called the second fundamental form of M in N. Here T^*M denotes the cotangent bundle of M. The mean curvature vector field of M in N is defined by $H = (\operatorname{trace} B)/m \in C(NM)$.

Furthermore, if $\eta \in C(NM)$, then

$$\nabla^\varphi_X \eta = -d\varphi(A_\eta(X)) + \nabla^\perp_X \eta, \quad \forall\, X \in C(TM),$$

where $A_\eta \in C(T^*M \otimes TM)$ is called the shape operator of M in N in the direction of η, and $\nabla^\perp$ is the induced connection in the normal bundle. Moreover, $\langle B(X, Y), \eta \rangle = \langle A_\eta(X), Y \rangle$, for all $X, Y \in C(TM)$, $\eta \in C(NM)$. In the case of hypersurfaces, we denote $f = \operatorname{trace} A$, where $A = A_\eta$ and η is the unit normal vector field, and we have $H = (f/m)\eta$; f is the *(m times) mean curvature function.*

A submanifold M of N is called PMC if A_H is parallel in the normal bundle, and CMC if $|H|$ is constant.

When confusion is unlikely, locally, we identify M with its image through φ, X with $d\varphi(X)$ and $\nabla^\varphi_X d\varphi(Y)$ with $\nabla^N_X Y$. With these identifications in mind, we write

$$\nabla^N_X Y = \nabla_X Y + B(X, Y),$$

and

$$\nabla^N_X \eta = -A_\eta(X) + \nabla^\perp_X \eta.$$

If $\operatorname{div} S_2 = 0$ for a submanifold M in N, then M is called *biconservative.* Thus, M is biconservative if and only if the tangent part of its bitension field vanishes.

We have the following characterisation theorem of biharmonic submanifolds obtained by splitting the bitension field in the tangent and normal part.

Theorem 2.3. *A submanifold M^m of a Riemannian manifold N^n is biharmonic if and only if*

$$\operatorname{trace} A_{\nabla^{\perp}_{\cdot} H}(\cdot) + \operatorname{trace} \nabla A_H + \operatorname{trace} \left(R^N(\cdot, H)\cdot\right)^T = 0$$

and

$$\Delta^{\perp} H + \operatorname{trace} B\left(\cdot, A_H(\cdot)\right) + \operatorname{trace} \left(R^N(\cdot, H)\cdot\right)^{\perp} = 0,$$

where $\Delta^{\perp}$ is the Laplacean in the normal bundle.

Various forms of the above result were obtained in [10, 18, 21]. From here we deduce some characterisation formulas for the biconservativity.

Corollary 2.4. *Let M^m be a submanifold of a Riemannian manifold N^n. Then M is a biconservative submanifold if and only if:*

(1) $\operatorname{trace} A_{\nabla^{\perp}_{\cdot} H}(\cdot) + \operatorname{trace} \nabla A_H + \operatorname{trace} \left(R^N(\cdot, H)\cdot\right)^T = 0$;
(2) $\frac{m}{2} \operatorname{grad}\left(|H|^2\right) + 2 \operatorname{trace} A_{\nabla^{\perp}_{\cdot} H}(\cdot) + 2 \operatorname{trace} \left(R^N(\cdot, H)\cdot\right)^T = 0$;
(3) $2 \operatorname{trace} \nabla A_H - \frac{m}{2} \operatorname{grad}\left(|H|^2\right) = 0$.

The following properties are immediate.

Proposition 2.5. *Let M^m be a submanifold of a Riemannian manifold N^n. If $\nabla A_H = 0$ then M is biconservative.*

Proposition 2.6. *Let M^m be a submanifold of a Riemannian manifold N^n. Assume that N is a space form, i.e., it has constant sectional curvature, and M is PMC. Then M is biconservative.*

Proposition 2.7 ([2])**.** *Let M^m be a submanifold of a Riemannian manifold N^n. Assume that M is pseudo-umbilical, i.e., $A_H = |H|^2 I$, and $m \neq 4$. Then M is CMC.*

If we consider the particular case of hypersurfaces, then Theorem 2.3 becomes

Theorem 2.8 ([2, 23])**.** *If M^m is a hypersurface in a Riemannian manifold N^{m+1}, then M is biharmonic if and only if*

$$2A(\operatorname{grad} f) + f \operatorname{grad} f - 2f \left(\operatorname{Ricci}^N(\eta)\right)^T = 0,$$

and

$$\Delta f + f|A|^2 - f \operatorname{Ricci}^N(\eta, \eta) = 0,$$

where η is the unit normal vector field of M in N.

Corollary 2.9. *A hypersurface M^m in a space form $N^{m+1}(c)$ is biconservative if and only if*

$$A(\operatorname{grad} f) = -\frac{f}{2} \operatorname{grad} f.$$

Corollary 2.10. *Any CMC hypersurface in $N^{m+1}(c)$ is biconservative.*

3. Local, intrinsic characterisation of biconservative surfaces

We are interested to study biconservative surfaces which are not CMC. We will study them from a local point of view and then from a global point of view. While by "local" we will mean the biconservative surfaces $\varphi : M^2 \to N^3(c)$ with $f > 0$ and $\operatorname{grad} f \neq 0$ at any point of M, by "global" we will mean the *complete* biconservative surfaces $\varphi : M^2 \to N^3(c)$ with $f > 0$ at any point of M and $\operatorname{grad} f \neq 0$ at any point of an open and dense subset of M.

In this section, we consider the local problem, i.e., we consider $\varphi : M^2 \to N^3(c)$ a biconservative surface and we assume that $f > 0$ and $\operatorname{grad} f \neq 0$ at any point of M. Let $X_1 = (\operatorname{grad} f)/|\operatorname{grad} f|$ and X_2 two vector fields such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$. In particular, we obtain that M is parallelizable. If we denote by $\lambda_1 \leq \lambda_2$ the eigenvalues functions of the shape operator A, since $A(X_1) = -(f/2)X_1$ and $\operatorname{trace} A = f$, we get $\lambda_1 = -f/2$ and $\lambda_2 = 3f/2$. Thus the matrix of A with respect to the (global) orthonormal frame field $\{X_1, X_2\}$ is

$$A = \begin{pmatrix} -\frac{f}{2} & 0 \\ 0 & \frac{3f}{2} \end{pmatrix}.$$

We denote by K the Gaussian curvature and, from the Gauss equation $K = c + \det A$, we obtain

$$f^2 = \frac{4}{3}(c - K). \tag{3.1}$$

Thus $c - K > 0$ on M.

From the way how X_1 and X_2 were defined, we find that

$$\operatorname{grad} f = (X_1 f)\, X_1 \quad \text{and} \quad X_2 f = 0.$$

Using the connection 1-forms, the Codazzi equation and then the extrinsic and intrinsic expression for the Gaussian curvature, we obtain the next result which shows that the mean curvature function of a non-CMC biconservative surface must satisfy a second-order partial differential equation. More precisely, we have the following theorem.

Theorem 3.1 ([8]). *Let $\varphi : M^2 \to N^3(c)$ a biconservative surface with $f > 0$ and $\operatorname{grad} f \neq 0$ at any point of M. Then we have*

$$f\Delta f + |\operatorname{grad} f|^2 + \frac{4}{3}cf^2 - f^4 = 0, \tag{3.2}$$

where Δ is the Laplace-Beltrami operator on M.

In fact, we can see that around any point of M there exists $(U; u, v)$ local coordinates such that $f = f(u, v) = f(u)$ and (3.2) is equivalent to

$$ff'' - \frac{7}{4}\left(f'\right)^2 - \frac{4}{3}cf^2 + f^4 = 0, \tag{3.3}$$

i.e., f must satisfy a *second-order ordinary differential equation.*

Indeed, let $p_0 \in M$ be an arbitrary fixed point of M and let $\gamma = \gamma(u)$ be an integral curve of X_1 with $\gamma(0) = p_0$. Let ϕ the flow of X_2 and $(U; u, v)$ local coordinates with $p_0 \in U$ such that

$$X(u, v) = \phi_{\gamma(u)}(v) = \phi(\gamma(u), v).$$

We have

$$X_u(u, 0) = \gamma'(u) = X_1(\gamma(u)) = X_1(u, 0)$$

and

$$X_v(u,v) = \phi'_{\gamma(u)}(v) = X_2\left(\phi_{\gamma(u)}(v)\right) = X_2(u,v).$$

If we write the Riemannian metric g on M in local coordinates as

$$g = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2,$$

we get $g_{22} = |X_v|^2 = |X_2|^2 = 1$, and X_1 can be expressed with respect to X_u and X_v as

$$X_1 = \frac{1}{\sigma}\left(X_u - g_{12}X_v\right) = \sigma \operatorname{grad} u,$$

where $\sigma = \sqrt{g_{11} - g_{12}^2} > 0$, $\sigma = \sigma(u,v)$.

Let $f \circ X = f(u,v)$. Since $X_2 f = 0$, we find that

$$f(u,v) = f(u,0) = f(u), \quad \forall (u,v) \in U.$$

It can be proved that

$$[X_1, X_2] = \frac{3\left(X_1 f\right)}{4f} X_2,$$

thus $X_2X_1f = X_1X_2f - [X_1,X_2]\,f = 0$.

On the other hand we have

$$\begin{aligned} X_2X_1f = {} & X_v\left(\tfrac{1}{\sigma}f'\right) = X_v\left(\tfrac{1}{\sigma}\right)f' \\ = {} & 0 \end{aligned} \quad . \tag{3.4}$$

We recall that

$$\operatorname{grad} f = \left(X_1 f\right) X_1 = \left(\frac{1}{\sigma}f'\right) X_1 \neq 0$$

at any point of U, thus $f' \neq 0$ at any point of U. Therefore, from (3.4), $X_v\left(1/\sigma\right) = 0$, i.e., $\sigma = \sigma(u)$. Since $g_{11}(u,0) = 1$, and $g_{12}(u,0) = 0$, we have $\sigma = 1$, i.e.,

$$X_1 = X_u - g_{12}X_v = \operatorname{grad} u. \tag{3.5}$$

In [8] it was found an equivalent expression for (3.2), i.e.,

$$\left(X_1X_1f\right)f = \frac{7}{4}\left(X_1f\right)^2 + \frac{4c}{3}f^2 - f^4.$$

Therefore, using (3.5), the relation (3.2) is equivalent to (3.3).

Remark 3.2. If $\varphi : M^2 \to N^3(c)$ is a non CMC biharmonic surface, then, there exists an open subset U such that $f > 0$ and $\operatorname{grad} f \neq 0$ at any point of U, and f satisfies the following system

$$\begin{cases} \Delta f = f\left(2c - |A|^2\right) \\ \\ A(\operatorname{grad} f) = -\frac{f}{2}\operatorname{grad} f. \end{cases}$$

As we have seen, this system implies

$$\begin{cases} \Delta f = f\left(2c - |A|^2\right) \\ \\ f\Delta f + |\operatorname{grad} f|^2 + \frac{4}{3}cf^2 - f^4 = 0. \end{cases}$$

which, in fact, is a ODE system. We get

$$\begin{cases} ff'' - \frac{3}{4}\left(f'\right)^2 + 2cf^2 - \frac{5}{2}f^4 = 0 \\ \\ ff'' - \frac{7}{4}\left(f'\right)^2 - \frac{4}{3}cf^2 + f^4 = 0 \end{cases} \tag{3.6}$$

As an immediate consequence we obtain

$$\left(f'\right)^2+\frac{10}{3}cf^2-\frac{7}{2}f^4=0,$$

and combining it with the prime integral

$$\left(f'\right)^2=2f^4-8cf^2+df^{3/2}$$

of the first equation from (3.6), where $d\in\mathbb{R}$ is a constant, we obtain

$$\frac{3}{2}f^{5/2}+\frac{14}{3}cf^{1/2}-d=0.$$

If we denote $\tilde{f}=f^{1/2}$, we get $\frac{3}{2}\tilde{f}^5+\frac{14}{3}c\tilde{f}-d=0$. Thus, $\tilde{f}$ satisfies a polynomial equation with constant coefficients, so $\tilde{f}$ has to be a constant and then, f is a constant, i.e., $\operatorname{grad} f=0$ on U (in fact, the constant has to be zero). Therefore, we have a contradiction (see [9, 11] for $c=0$ and [6, 7], for $c=\pm 1$)

We also can note that the relation (3.2), which is an extrinsic relation, together with (3.1), allows us to find an *intrinsic relation* that (M,g) must satisfy. More precisely, the Gaussian curvature of M has to satisfy

$$(c-K)\Delta K-|\operatorname{grad} K|^2-\frac{8}{3}K(c-K)^2=0, \tag{3.7}$$

and the hypotheses $f>0$ and $\operatorname{grad} f\neq 0$, at any point, are equivalent with the intrinsic condition $c-K>0$ and $\operatorname{grad} K\neq 0$.

The formula (3.7) is very similar to the Ricci condition. Further, we will briefly recall the Ricci problem. Given an abstract surface $\left(M^2,g\right)$, we want to find the conditions that has to be satisfied by M such that, locally, it admits a minimal embedding in $N^3(c)$. It was proved that if $\left(M^2,g\right)$ is an abstract surface such that $c-K>0$ at any point of M, where $c\in\mathbb{R}$ is a constant, then, locally, it admits a minimal embedding in $N^3(c)$ if and only if

$$(c-K)\Delta K-|\operatorname{grad} K|^2-4K(c-K)^2=0. \tag{3.8}$$

Condition (3.8) is called the *Ricci condition with respect cu* c, or simply the *Ricci condition*. If (3.8) holds, then M admits a one-parameter family of minimal embeddings in $N^3(c)$.

We can see that the relations (3.7) and (3.8) are very similar and, in [12] the authors studied the link between them. Thus, for $c=0$, it was proved that if we consider a surface $\left(M^2,g\right)$ which satisfies (3.7) and $K<0$, then there exists a very simple conformal transformation of the metric g such that $\left(M^2,\sqrt{-K}g\right)$ satisfies (3.8). A similar result was proved also for $c\neq 0$, but is this case, the conformal factor has a complicated expression (and it is not enough to impose that $\left(M^2,g\right)$ satisfy (3.7), but we need the stronger hypothesis of it admitting a biconservative immersion in $N^3(c)$).

Unfortunately, the condition (3.7) does not imply the existence of a biconservative immersion in $N^3(c)$, as in the minimal case. We need a stronger condition. It was obtained the following local, intrinsic characterisation theorem.

Theorem 3.3 ([12])**.** *Let* $\left(M^2,g\right)$ *be an abstract surface and* $c\in\mathbb{R}$ *a constant. Then, locally,* M *can be isometrically embedded in a space form* $N^3(c)$ *as a biconservative surface with positive mean curvature having the gradient different from zero at any point* $p\in M$ *if and only if the Gaussian curvature* K *satisfies* $c-K(p)>0$*,*

$(\operatorname{grad} K)(p) \neq 0$, *for any point* $p \in M$, *and its level curves are circles in* M *with curvature*

$$\kappa = \frac{3|\operatorname{grad} K|}{8(c-K)}.$$

Remark 3.4. If the surface M in Theorem 3.3 is simply connected, then the theorem holds globally, but, in this case, instead of a local isometric embedding we have a global isometric immersion.

We remark that unlike the minimal immersions case, if M satisfies the hypotheses from Theorem 3.3, then there exists an *unique* biconservative immersion in $N^3(c)$ (up to an isometry of $N^3(c)$), and not a one-parameter family.

The characterisation theorem can be rewritten in an equivalent way, as below.

Theorem 3.5. *Let* $\left(M^2, g\right)$ *be an abstract surface with Gaussian curvature* K *satisfying* $c - K(p) > 0$ *and* $(\operatorname{grad} K)(p) \neq 0$ *at any point* $p \in M$, *where* $c \in \mathbb{R}$ *is a constant. Let* $X_1 = (\operatorname{grad} K)/|\operatorname{grad} K|$ *and* $X_2 \in C(TM)$ *be two vector fields on* M *such that* $\{X_1(p), X_2(p)\}$ *is a positively oriented basis at any point of* $p \in M$. *Then, the following conditions are equivalent:*

(a) *the level curves of* K *are circles in* M *with constant curvature*

$$\kappa = \frac{3X_1K}{8(c-K)};$$

(b)

$$X_2\left(X_1K\right) = 0 \quad \text{and} \quad \nabla_{X_2}X_2 = \frac{-3X_1K}{8(c-K)}X_1;$$

(c) *locally, the metric* g *can be written as* $g = (c-K)^{-3/4}\left(du^2 + dv^2\right)$, *where* (u, v) *are local coordinates positively oriented,* $K = K(u)$, *and* $K' > 0$;

(d) *locally, the metric* g *can be written as* $g = e^{2\varphi}\left(du^2 + dv^2\right)$, *where* (u, v) *are local coordinates positively oriented, and* $\varphi = \varphi(u)$ *satisfies the equation*

$$\varphi'' = e^{-2\varphi/3} - ce^{2\varphi} \tag{3.9}$$

and the condition $\varphi' > 0$; *moreover, the solutions of the above equation,* $u = u(\varphi)$, *are*

$$u = \int_{\varphi_0}^{\varphi} \frac{d\tau}{\sqrt{-3e^{-2\tau/3} - ce^{2\tau} + a}} + u_0,$$

where φ *is in some open interval* I *and* $a, u_0 \in \mathbb{R}$ *are constants;*

(e) *locally, the metric* g *can be written as* $g = e^{2\varphi}\left(du^2 + dv^2\right)$, *where* (u, v) *are local coordinates positively oriented, and* $\varphi = \varphi(u)$ *satisfies the equation*

$$3\varphi''' + 2\varphi'\varphi'' + 8ce^{2\varphi}\varphi' = 0 \tag{3.10}$$

and the conditions $\varphi' > 0$ *and* $c + e^{-2\varphi}\varphi'' > 0$; *moreover, the solutions of the above equation,* $u = u(\varphi)$, *are*

$$u = \int_{\varphi_0}^{\varphi} \frac{d\tau}{\sqrt{-3be^{-2\tau/3} - ce^{2\tau} + a}} + u_0,$$

where φ *is in some open interval* I *and* $a, b, u_0 \in \mathbb{R}$ *are constants,* $b > 0$.

The proof follows by direct computations and by using Remark 4.3 in [12] and Proposition 3.4 in [20].

Remark 3.6. From the above theorem we have the following remarks.

(i) If the condition (a) is satisfied, i.e., the integral curves of X_2 are circles in M with a precise constant curvature, then the integral curves of X_1 are geodesics of M;

(ii) If the condition (c) is satisfied, then K has to be a solution of the equation

$$3K''(c-K)+3\left(K'\right)^2+8K(c-K)^{5/4}=0;$$

(iii) Let $\varphi=\varphi(u)$ be a solution of the equation (3.10). We consider the change of coordinates

$$(u,v)=\left(\alpha\tilde{u}+\beta,\alpha\tilde{v}+\beta\right),$$

where $\alpha\in\mathbb{R}$ is a positive constant and $\beta\in\mathbb{R}$, and define

$$\phi=\varphi\left(\alpha\tilde{u}+\beta\right)+\log\alpha.$$

Then $g=e^{2\phi}\left(d\tilde{u}^2+d\tilde{v}^2\right)$ and ϕ also satisfies the equation (3.10). If $\varphi=\varphi(u)$ satisfies the first integral

$$\varphi''=be^{-2\varphi/3}-ce^{2\varphi},$$

where $b>0$, then, for $\alpha=b^{-3/8}$, $\phi=\phi\left(\tilde{u}\right)$ satisfies

$$\phi''=e^{-2\phi/3}-ce^{2\phi}.$$

From here, as the classification is done up to isometries, we note that the parameter b in the solution of (3.10) is not essential and only the parameter a counts.

(iv) If $c=0$, we note that if φ is a solution of (3.10), then also $\varphi+\text{constant}$ is a solution of the same equation, i.e, the condition (a) from the Theorem 3.5 is invariant under the homothetic tranformations of the metric g. Then, we see that the equation (3.10) is invariant under the affine change of parameter $u=\alpha\tilde{u}+\beta$, where $\alpha>0$. Therefore, we must solve the equation (3.10) up to this change of parameter and an additive constant of the solution φ. The additive constant will be the parameter that counts.

An abstract surface $\left(M^2,g\right)$ that admits a biconservative immersion in $N^3(c)$ is also called *biconservative surface with respect to* c, or simply *biconservative surface*.

In the case $c=0$, the solutions of the equation (3.10), are explicitly determined in the next proposition.

Proposition 3.7 ([20])**.** *The solutions of the equation*

$$3\varphi'''+2\varphi'\varphi''=0$$

which satisfy the conditions $\varphi'>0$ and $\varphi''>0$, up to affine transformations of the parameter with $\alpha>0$, are given by

$$\varphi(u)=3\log(\cosh u)+\text{constant},\qquad u>0.$$

We note that, in the case $c=0$, we have a one-parameter family of solutions of equation (3.10), i.e., $g_{C_0}=C_0(\cosh u)^6\left(du^2+dv^2\right)$, C_0 being a positive constant.

If $c\neq0$, then we can not determine explicitly $\varphi=\varphi(u)$. Another way to see that in the case $c\neq0$ we have only a one-parameter family of solutions of equation (3.10) is to rewrite the metric g in certain non-isothermal coordinates.

Further, we will consider only the case $c=1$ and we have the next result.

Proposition 3.8 ([20])**.** *Let $\left(M^2,g\right)$ be an abstract surface with $g=e^{2\varphi(u)}(du^2+dv^2)$, where $u=u(\varphi)$ satisfies*

$$u=\int_{\varphi_0}^{\varphi}\frac{d\tau}{\sqrt{-3be^{-2\tau/3}-e^{2\tau}+a}}+u_0,$$

where φ is in some open interval I, $a, b \in \mathbb{R}$ are positive constants, and $u_0 \in \mathbb{R}$ is a constant. Then $\left(M^2, g\right)$ is isometric to

$$\left(D_{C_1}, g_{C_1} = \frac{3}{\xi^2\left(-\xi^{8/3} + 3C_1\xi^2 - 3\right)} d\xi^2 + \frac{1}{\xi^2} d\theta^2\right),$$

where $D_{C_1} = (\xi_{01}, \xi_{02}) \times \mathbb{R}$, $C_1 \in \left(4/\left(3^{3/2}\right), \infty\right)$ is a positive constant, and ξ_{01} and ξ_{02} are the positive vanishing points of $-\xi^{8/3} + 3C_1\xi^2 - 3$, with $0 < \xi_{01} < \xi_{02}$.

Remark 3.9. Let us consider

$$\left(D_{C_1}, g_{C_1} = \frac{3}{\xi^2\left(-\xi^{8/3} + 3C_1\xi^2 - 3\right)} d\xi^2 + \frac{1}{\xi^2} d\theta^2\right)$$

and

$$\left(D_{C_1'}, g_{C_1'} = \frac{3}{\tilde{\xi}^2\left(-\tilde{\xi}^{8/3} + 3C_1'\tilde{\xi}^2 - 3\right)} d\tilde{\xi}^2 + \frac{1}{\tilde{\xi}^2} d\tilde{\theta}^2\right).$$

The surfaces (D_{C_1}, g_{C_1}) and $\left(D_{C_1'}, g_{C_1'}\right)$ are isometric if and only if $C = C_1'$ and the isometry is $\Theta(\xi, \theta) = (\xi, \pm\theta + \text{constant})$. Therefore, we have a one-parameter family of surfaces.

Remark 3.10. We note that the Gaussian curvature of (D_{C_1}, g_{C_1}) does not depend on C_1. More precisely,

$$K_{C_1}(\xi, \theta) = -\frac{1}{9}\xi^{8/3} + 1.$$

But, if we change further the coordinates $(\xi, \theta) = \left(\xi_{01} + \tilde{\xi}\left(\xi_{02} - \xi_{01}\right), \tilde{\theta}\right)$, then we "fix" the domain, i.e., (D_{C_1}, g_{C_1}) is isometric to $((0,1), \tilde{g}_{C_1})$ and C_1 appears in the expression of $K_{C_1}\left(\tilde{\xi}, \tilde{\theta}\right)$.

4. Complete biconservative surfaces in $\mathbb{R}^3$

In this section we consider the global problem and construct complete biconservative surfaces in $\mathbb{R}^3$ with $f > 0$ everywhere and $\operatorname{grad} f \neq 0$ at any point of an open dense subset. Or, from intrinsic point of view, we construct a complete abstract surface $\left(M^2, g\right)$ with $K < 0$ everywhere and $\operatorname{grad} K \neq 0$ at any point of an open dense subset of M, that admits a biconservative immersion in $\mathbb{R}^3$.

First, we recall a local extrinsic result about biconservative surfaces which says that a biconservative surface in $\mathbb{R}^3$ is, locally, a surface of revolution.

Theorem 4.1 ([14])**.** *Let M^2 be a biconservative surface in $\mathbb{R}^3$ with $f(p) > 0$ and $(\operatorname{grad} f)(p) \neq 0$ for any $p \in M$. Then, locally, M^2 is a surface of revolution, and the curvature $\kappa = \kappa(u)$ of the profile curve $\sigma = \sigma(u)$, $|\sigma'(u)| = 1$, is a positive solution of the following ODE*

$$\kappa''\kappa = \frac{7}{4}\left(\kappa'\right)^2 - 4\kappa^4.$$

In [8] it was found the local explicit parametric equation of a biconservative surface in $\mathbb{R}^3$.

Theorem 4.2. [8] *Let M^2 be a biconservative surface in $\mathbb{R}^3$ with $f(p) > 0$ and $(\operatorname{grad} f)(p) \neq 0$ for any $p \in M$. Then, locally, the surface can be parameterized by*

$$X_{\tilde{C}_0}(\rho, v) = \left(\rho\cos v, \rho\sin v, u_{\tilde{C}_0}(\rho)\right),$$

where

$$u_{\tilde{C}_0}(\rho)=\frac{3}{2\tilde{C}_0}\left(\rho^{1/3}\sqrt{\tilde{C}_0\rho^{2/3}-1}+\frac{1}{\sqrt{\tilde{C}_0}}\log\left(\sqrt{\tilde{C}_0}\rho^{1/3}+\sqrt{\tilde{C}_0\rho^{2/3}-1}\right)\right)$$

with $\tilde{C}_0$ *a positive constant and* $\rho\in\left(\tilde{C}_0^{-3/2},\infty\right)$.

We note that any two such surfaces are not locally isometric, so we have a one-parameter family of biconservative surfaces in $\mathbb{R}^3$. These surfaces are not complete.

We denote be $S_{\tilde{C}_0}$ the image of $X_{\tilde{C}_0}$. The "boundary" of $S_{\tilde{C}_0}$, i.e., $\overline{S}_{\tilde{C}_0}\setminus S_{\tilde{C}_0}$, is the circle $\left(\tilde{C}_0^{-3/2}\cos v,\tilde{C}_0^{-3/2}\sin v,0\right)$, which lies in the xOy plane. At a boundary point, the tangent plane to the closure $\overline{S}_{\tilde{C}_0}$ of $S_{\tilde{C}_0}$ is parallel to Oz. Moreover, along the boundary, the mean curvature function is constant $f_{\tilde{C}_0}=\left(2\tilde{C}_0^{3/2}\right)/3$ and $\operatorname{grad} f_{\tilde{C}_0}=0$.

Thus, we can expect to "glue" along the boundary two biconservative surfaces of type $S_{\tilde{C}_0}$ corresponding to the same $\tilde{C}_0$ (the two constants have to be the same) and symmetric to each other, at the level of C^∞ smoothness.

In fact, it was proved that we can glue two biconservative surfaces $S_{\tilde{C}_0}$ and $S_{\tilde{C}_0'}$, at the level of C^∞ smoothness, only along the boundary and, in this case $\tilde{C}_0=\tilde{C}_0'$.

Proposition 4.3 ([20, 19])**.** *If we consider the symmetry of* $\operatorname{Graf} u_C$*, with respect to the* $O\rho(=Ox)$ *axis, we get a smooth, complete, biconservative surface* $\tilde{S}_{\tilde{C}_0}$ *in* $\mathbb{R}^3$*. Moreover, its mean curvature function* $f_{\tilde{C}_0}$ *is positive and* $\operatorname{grad} f_{\tilde{C}_0}$ *is different from zero at any point of an open dense subset of* $\tilde{S}_{\tilde{C}_0}$*.*

Remark 4.4. The profile curve $\sigma_{\tilde{C}_0}=\left(\rho,0,u_{\tilde{C}_0}(\rho)\right)\equiv\left(\rho,u_{\tilde{C}_0}(\rho)\right)$ can be reparameterized as

(4.1)
$$\begin{aligned}\sigma_{\tilde{C}_0}(\theta)=&\left(\sigma^1_{\tilde{C}_0}(\theta),\sigma^2_{\tilde{C}_0}(\theta)\right)\\ =&\tilde{C}_0^{-3/2}\left((\theta+1)^{3/2},\tfrac{3}{2}\left(\sqrt{\theta^2+\theta}+\log\left(\sqrt{\theta}+\sqrt{\theta+1}\right)\right)\right),\qquad \theta>0,\end{aligned}$$

and now $X_{\tilde{C}_0}=X_{\tilde{C}_0}(\theta,v)$.

Proposition 4.5. *The homothety of* $\mathbb{R}^3$*,* $(x,y,z)\to\tilde{C}_0(x,y,z)$*, renders* $\tilde{S}_1$ *onto* $\tilde{S}_{\tilde{C}_0^{-2/3}}$*.*

In [20], it were also found the complete biconservative surfaces in $\mathbb{R}^3$ with $f>0$ at any point and $\operatorname{grad} f\neq 0$ at any point of an open dense subset, but there the idea was to use the intrinsic characterization of the biconservative surfaces. More precisely, we have the next global result.

Theorem 4.6 ([20])**.** *Let* $\left(\mathbb{R}^2,g_{C_0}=C_0\left(\cosh u\right)^6\left(du^2+dv^2\right)\right)$ *be a surface, where* $C_0\in\mathbb{R}$ *is a positive constant. Then we have:*

(a) *the metric on* $\mathbb{R}^2$ *is complete;*
(b) *the Gaussian curvature is given by*

$$K_{C_0}(u,v)=K_{C_0}(u)=-\frac{3}{C_0\left(\cosh u\right)^8}<0,\quad K'_{C_0}(u)=\frac{24\sinh u}{C_0\left(\cosh u\right)^9},$$

and therefore $\operatorname{grad} K_{C_0}\neq 0$ *at any point of* $\mathbb{R}^2\setminus Ov$*;*

(c) *the immersion* $\varphi_{C_0} : \left(\mathbb{R}^2, g_{C_0}\right) \to \mathbb{R}^3$ *given by*

$$\varphi_{C_0}(u,v) = \left(\sigma^1_{C_0}(u)\cos 3v, \sigma^1_{C_0}(u)\sin 3v, \sigma^2_{C_0}(u)\right)$$

is biconservative in $\mathbb{R}^3$, *where*

$$\sigma^1_{C_0}(u) = \frac{\sqrt{C_0}}{3}\left(\cosh u\right)^3, \quad \sigma^2_{C_0}(u) = \frac{\sqrt{C_0}}{2}\left(\frac{1}{2}\sinh 2u + u\right), \qquad u \in \mathbb{R}.$$

Sketch of the proof. The first two items follow by standard arguments. For the last part, we note that choosing $\tilde{C}_0 = (9/C_0)^{1/3}$ in (4.1) and using the change of coordinates $(\theta, v) = \left((\sinh u)^2, 3v\right)$, where $u > 0$, the metric induced by $X_{(9/C_0)^{1/3}}$ coincides with g_{C_0}. Then, we define φ_{C_0} as: for $u > 0$, $\varphi_{C_0}(u,v)$ is obtained by rotating the profile curve

$$\sigma^+_{(9/C_0)^{1/3}}(u) = \sigma_{(9/C_0)^{1/3}}(u) = \left(\sigma^1_{(9/C_0)^{1/3}}(u), \sigma^2_{(9/C_0)^{1/3}}(u)\right),$$

and for $u < 0$, $\varphi_{C_0}(u,v)$ is obtained by rotating the profile curve

$$\sigma^-_{(9/C_0)^{1/3}}(u) = \left(\sigma^1_{(9/C_0)^{1/3}}(-u), -\sigma^2_{(9/C_0)^{1/3}}(-u)\right).$$

$\square$

Theorem 4.7. *Let* $\left(\mathbb{R}^2, g_{C_0}\right)$ *be a biconservative surface with respect to* $c = 0$. *Then* $\left(\mathbb{R}^2, \sqrt{-K} g_{C_0}\right)$ *satisfies the Ricci condition and can be minimally immersed in* $\mathbb{R}^3$ *as a helicoid.*

Concerning the biharmonic surfaces in $\mathbb{R}^3$ we have the following non-existence result.

Theorem 4.8 ([9, 11])**.** *There exists no proper biharmonic surface in* $\mathbb{R}^3$.

5. Complete biconservative surfaces in $\mathbb{S}^3$

As in the previous section, we consider the global problem for biconservative surfaces in $\mathbb{S}^3$, i.e., our aim is to construct complete biconservative surfaces in $\mathbb{S}^3$ with $f > 0$ and $\operatorname{grad} f \neq 0$ at any point of an open and dense subset.

We start with the following local, extrinsic result.

Theorem 5.1 ([8])**.** *Let* M^2 *be a biconservative surface in* $\mathbb{S}^3$ *with* $f(p) > 0$ *şi* $(\operatorname{grad} f)(p) \neq 0$ *at any point* $p \in M$. *Then, locally, the surface, viewed in* $\mathbb{R}^4$, *can be parameterized by*

$$Y_{\tilde{C}_1}(u,v) = \sigma(u) + \frac{4\kappa(u)^{-3/4}}{3\sqrt{\tilde{C}_1}}\left(\overline{f}_1(\cos v - 1) + \overline{f}_2 \sin v\right),$$

where $\tilde{C}_1 \in \left(64/\left(3^{5/4}\right), \infty\right)$ *is a positive constant;* $\overline{f}_1, \overline{f}_2 \in \mathbb{R}^4$ *are two constant orthonormal vectors;* $\sigma(u)$ *is a curve parameterized by arclength that satisfies*

$$\langle \sigma(u), \overline{f}_1\rangle = \tfrac{4\kappa(u)^{-3/4}}{3\sqrt{\tilde{C}_1}}, \quad \langle \sigma(u), \overline{f}_2\rangle = 0,$$

and, as a curve in $\mathbb{S}^2$, *its curvature* $\kappa = \kappa(u)$ *is a positive non constant solution of the following ODE*

$$\kappa''\kappa = \frac{7}{4}\left(\kappa'\right)^2 + \frac{4}{3}\kappa^2 - 4\kappa^4$$

such that

$$\left(\kappa'\right)^2 = -\frac{16}{9}\kappa^2 - 16\kappa^4 + \tilde{C}_1\kappa^{7/2}.$$

Remark 5.2. The constant $\tilde{C}_1$ determines uniquely the curvature κ, up to a translation of u, and then κ, $\overline{f}_1$ and $\overline{f}_2$ determines uniquely the curve σ.

We consider $\overline{f}_1 = \overline{e}_3$ and $\overline{f}_2 = \overline{e}_4$ and change the coordinates (u, v) in (κ, v). Then, we get
(5.1)

$$\begin{aligned} Y_{\tilde{C}_1}(\kappa, v) = \ & \Bigg(\sqrt{1-\left(\tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa^{-3/4}\right)^2}\cos\mu(\kappa), \sqrt{1-\left(\tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa^{-3/4}\right)^2}\sin\mu(\kappa), \\ & \tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa^{-3/4}\cos v, \tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa^{-3/4}\sin v\Bigg), \end{aligned}$$

where $(\kappa, v) \in (\kappa_{01}, \kappa_{02}) \times \mathbb{R}$, κ_{01} and κ_{02} are positive solutions of

$$-\frac{16}{9}\kappa^2 - 16\kappa^4 + \tilde{C}_1\kappa^{7/2} = 0$$

and

$$\mu(\kappa) = \pm 108\int_{\kappa_0}^{\kappa} \frac{\tau^{3/4}\sqrt{\tilde{C}_1}}{\left(-16+9\tilde{C}_1\tau^{3/2}\right)\sqrt{9\tilde{C}_1\tau^{3/2}-16\left(1+9\tau^2\right)}}d\tau + c_0,$$

with $c_0 \in \mathbb{R}$ a constant and $\kappa_0 \in (\kappa_{01}, \kappa_{02})$. We note that an alternative expression for $Y_{\tilde{C}_1}$ was given in [13].

Remark 5.3. For simplicity, we choose $\kappa_0 = (3\tilde{C}_1/64)^2$.

If we denote $S_{\tilde{C}_1}$ the image of $Y_{\tilde{C}_1}$, then we note that the boundary of $S_{\tilde{C}_1}$ is made up from two circles and along the boundary, the mean curvature function is constant (two different constants) and its gradient vanishes.

Thus, we can expect to glue along the boundary two biconservative surfaces of type $S_{\tilde{C}_1}$, corresponding to the same $\tilde{C}_1$. In fact, if we want to glue two surfaces corresponding to $\tilde{C}_1$ and $\tilde{C}_1'$ along the boundary, then these constant have to coincide and there is no ambiguity concerning along which circle of the boundary we should glue the two pieces. But this process is not as clear as in $\mathbb{R}^3$ since we should repeat it infinitely many times.

Further, as in the $\mathbb{R}^3$ case, we change the point of view and use the intrinsic characterization of the biconservative surfaces in $\mathbb{S}^3$.

The surface (D_{C_1}, g_{C_1}) defined in Section 3 is note complete but it has the following properties.

Theorem 5.4 ([20]). *Consider (D_{C_1}, g_{C_1}). Then, we have*

(a) $K_{C_1}(\xi, \theta) = K(\xi, \theta)$,

$$1 - K(\xi,\theta) = \frac{1}{9}\xi^{8/3} > 0, \quad K'(\xi) = -\frac{8}{27}\xi^{5/3}$$

and grad $K \neq 0$ *at any point of* D_{C_1}*;*

(b) *the immersion* $\phi_{C_1} : (D_{C_1}, g_{C_1}) \to \mathbb{S}^3$ *given by*

$$\phi_{C_1}(\xi,\theta) = \left(\sqrt{1-\frac{1}{C_1\xi^2}}\cos\zeta(\xi), \sqrt{1-\frac{1}{C_1\xi^2}}\sin\zeta(\xi), \frac{\cos(\sqrt{C_1}\theta)}{\sqrt{C_1}\xi}, \frac{\sin(\sqrt{C_1}\theta)}{\sqrt{C_1}\xi}\right),$$

is biconservative in $\mathbb{S}^3$*, where*

$$\zeta(\xi) = \pm\int_{\xi_{00}}^{\xi} \frac{\sqrt{C_1}\tau^{4/3}}{(-1+C_1\tau^2)\sqrt{-\tau^{8/3}+3C_1\tau^2-3}}d\tau + c_1 = \pm\zeta_0(\xi) + c_1,$$

with $c_1 \in \mathbb{R}$ a constant and $\xi_{00} \in (\xi_{01}, \xi_{02})$.

Sketch of the proof. The first item follows by standard arguments. For the second item, we note that choosing $\tilde{C}_1 = 3^{1/4} \cdot 16C_1$ in (5.1) and using the change of coordinates $(\kappa, v) = \left(3^{-3/2}\xi^{4/3}, \left(3^{-1/8}\sqrt{C_1}\theta\right)/4\right)$, the metric induced by $Y_{3^{1/4}\cdot 16C_1}$ coincides with g_{C_1}.

Then, we define ϕ_{C_1} as

$$\phi_{C_1}(\xi, \theta) = Y_{3^{1/4}\cdot 16C_1}\left(3^{-3/2}\xi^{4/3}, \frac{3^{-1/8}\sqrt{C_1}\theta}{4}\right).$$

□

Remark 5.5. The limits $\lim_{\xi \searrow \xi_{01}} \zeta_0(\xi)$ and $\lim_{\xi \nearrow \xi_{02}} \zeta_0(\xi)$ are finite.

Remark 5.6. For simplicity, we choose $\xi_{00} = (9C_1/4)^{3/2}$.

Remark 5.7. The immersion ϕ_{C_1} depends on the sign $\pm$ and on the constant c_1 in the expression of ζ. As the classification is up to isometries of $\mathbb{S}^3$, the sign and the constant are not important, but they will play an important role in the gluing process.

The key idea in our construction is to notice that (D_{C_1}, g_{C_1}) is, locally and intrinsically, isometric to a surface of revolution in $\mathbb{R}^3$.

Theorem 5.8. *Let us consider (D_{C_1}, g_{C_1}) as above. Then (D_{C_1}, g_{C_1}) is the universal cover of the surface of revolution in $\mathbb{R}^3$ given by*

$$\psi_{C_1,C_1^*}(\xi, \theta) = \left(\chi(\xi)\cos\frac{\theta}{C_1^*}, \chi(\xi)\sin\frac{\theta}{C_1^*}, \nu(\xi)\right), \tag{5.2}$$

where $\chi(\xi) = C_1^*/\xi$,

$$\nu(\xi) = \pm\int_{\xi_{00}}^{\xi}\sqrt{\frac{3\tau^2 - (C_1^*)^2\left(-\tau^{8/3} + 3C_1\tau^2 - 3\right)}{\tau^4\left(-\tau^{8/3} + 3C_1\tau^2 - 3\right)}}d\tau + c_1^*, \tag{5.3}$$

$C_1^* \in \left(0, \left(C_1 - 4/3^{3/2}\right)^{-1/2}\right)$ is a positive constant and $c_1^* \in \mathbb{R}$ is constant.

Remark 5.9. The immersion ψ_{C_1,C_1^*} depends on the sign $\pm$ and on the constant c_1^* in the expression of ν. We denote by $S^{\pm}_{C_1,C_1^*,c_1^*}$ the image of ψ_{C_1,C_1^*}.

We fix C_1 and C_1^*, and alternating the sign and with appropriate choices of the constant c_1^*, we can construct a complete surface of revolution $\tilde{S}_{C_1,C_1^*}$ in $\mathbb{R}^3$ which on an open subset is locally isometric to (D_{C_1}, g_{C_1}). In fact, these choices of $+$ and $-$, and of the constants c_1^* are uniquely determined by the "first" choice of $+$, or of $-$, and of the constant c_1^*.

The profile curve of $\tilde{S}_{C_1,C_1^*}$ is the graph of a function $\chi \circ F$ depending on ν and defined on the whole Oz (or $O\nu$); here $F : \mathbb{R} \to [\xi_{01}, \xi_{02}]$ is a function at least of class C^3.

Theorem 5.10. *The surface of revolution given by*

$$\Psi_{C_1,C_1^*}(\nu, \theta) = \left((\chi\circ F)(\nu)\cos\frac{\theta}{C_1^*}, (\chi\circ F)(\nu)\sin\frac{\theta}{C_1^*}, \nu\right), \qquad (\nu, \theta) \in \mathbb{R}^2,$$

is complete and, on an open dense subset, it is locally isometric to (D_{C_1}, g_{C_1}). The induced metric is given by

$$g_{C_1,C_1^*}(\nu, \theta) = \frac{3F^2(\nu)}{3F^2(\nu) - (C_1^*)^2\left(-F^{8/3}(\nu) + 3C_1F^2(\nu) - 3\right)}d\nu^2 + \frac{1}{F^2(\nu)}d\theta^2,$$

$(\nu,\theta)\in\mathbb{R}^2$. Moreover, $\operatorname{grad} K\neq 0$ at any point of that open dense subset, and $1-K>0$ everywhere.

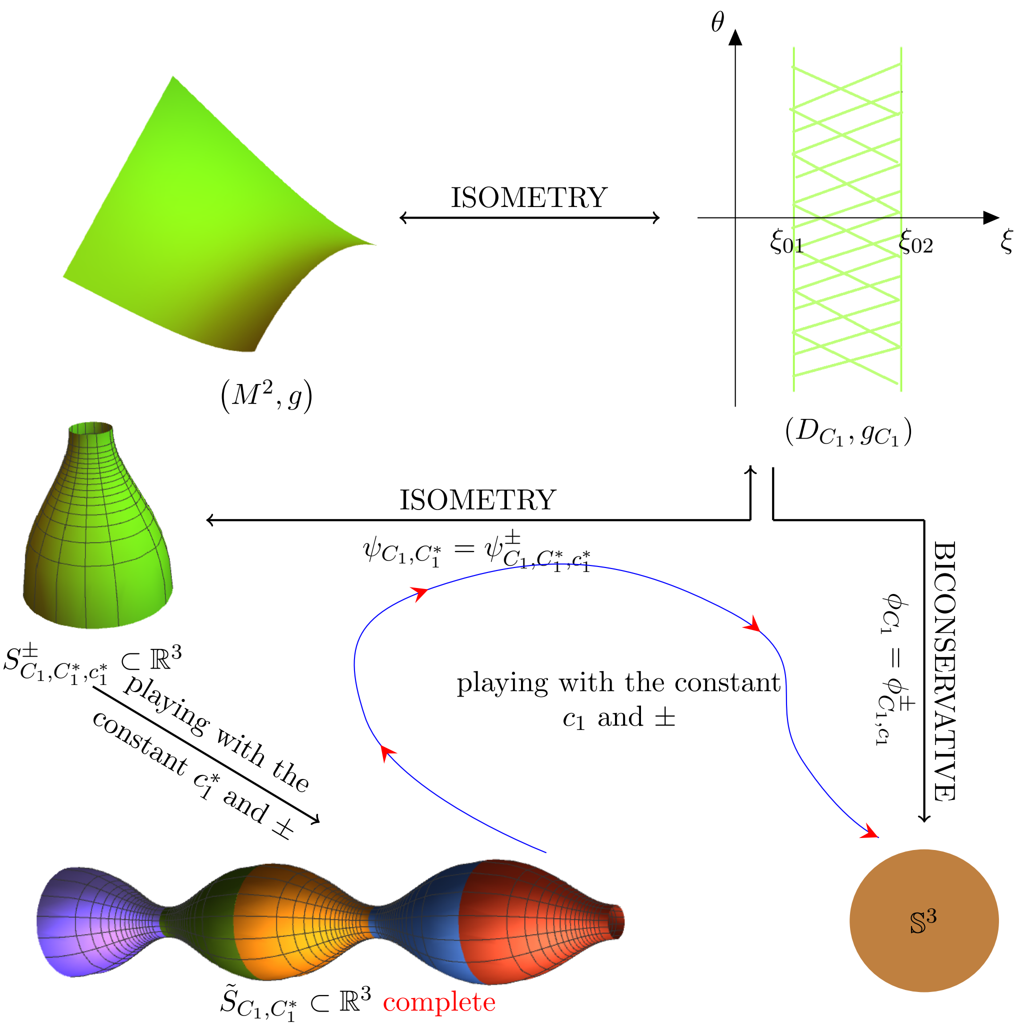
From Theorem 5.10 we easily get the following result.

Proposition 5.11. *The universal cover of the surface of revolution given by Ψ_{C_1,C_1^*} is $\mathbb{R}^2$ endowed with the metric g_{C_1,C_1^*}. It is complete, $1-K>0$ on $\mathbb{R}^2$ and, on an open dense subset, it is locally isometric to (D_{C_1},g_{C_1}) and $\operatorname{grad} K\neq 0$ at any point. Moreover any two $\left(\mathbb{R}^2,g_{C_1,C_1^*}\right)$ and $\left(\mathbb{R}^2,g_{C_1,C_1^{*\prime}}\right)$ are isometric.*

Appendix A

In the following, we illustrate the results from the case $c=0$, and the idea was to construct by symmetry a complete biconservative surface in $\mathbb{R}^3$, starting with a piece of a biconservative surface.

For the case $c=1$, the construction of a complete biconservative surface in $\mathbb{S}^3$ is not as simple as in the $\mathbb{R}^3$, but it can be illustrated by the next diagram.



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