ON BICONSERVATIVE SURFACES IN 3-DIMENSIONAL SPACE FORMS

DOREL FETCU, SIMONA NISTOR, AND CEZAR ONICIUC

Abstract. We consider biconservative surfaces $\left(M^2, g\right)$ in a space form $N^3(c)$, with mean curvature function f satisfying $f > 0$ and $\nabla f \neq 0$ at any point, and determine a certain Riemannian metric g_r on M such that $\left(M^2, g_r\right)$ is a Ricci surface in $N^3(c)$. We also obtain an intrinsic characterization of these biconservative surfaces.

1. Introduction

In the last few years, from the theory of *biharmonic submanifolds*, arised the study of *biconservative submanifolds* that imposed itself as a very promising and interesting research topic through papers like [4, 6, 11, 13, 14]. A *biharmonic map* $\psi : (M, g) \to (N, h)$ between two Riemannian manifolds is a critical point of the *bienergy functional*

$$E_2 : C^\infty(M, N) \to \mathbb{R}, \quad E_2(\psi) = \frac{1}{2}\int_M |\tau(\psi)|^2 \, dv,$$

where $\tau(\psi)$ is the tension field of ψ, and it is characterized by the vanishing of its *bitension field* $\tau_2(\psi)$. When $\psi : (M, g) \to (N, h)$ is a biharmonic isometric immersion, M is called a biharmonic submanifold of N.

Now, if $\psi : M \to (N, h)$ is a fixed map, then E_2 can be thought as a functional on the set of all Riemannian metrics on M. This new functional's critical points are Riemannian metrics determined by the vanishing of the *stress-energy tensor* S_2. This tensor satisfies

$$\operatorname{div} S_2 = -\langle \tau_2(\psi), d\psi \rangle.$$

If $\operatorname{div} S_2 = 0$ for a submanifold M in N, then M is called a biconservative submanifold and it is characterized by the fact that the tangent part of its bitension field vanishes.

In the case when the ambient space is a 3-dimensional space form $N^3(c)$, while surfaces with constant mean curvature (CMC surfaces) are trivially biconservative, the study of non-CMC biconservative surfaces is not trivial. The explicit local equations of these surfaces were obtained in [4] and [6]. Moreover, in [4] it is shown that the Gaussian curvature of a biconservative surface in a 3-dimensional space form satisfies a certain equation that seems to be very similar with that used by G. Ricci-Curbastro [17] in 1895 to characterize minimal surfaces in $\mathbb{R}^3$. As we will see in the following, we can use this property of biconservative surfaces to prove results similar to those in [10], [15], or [17], in this context.

2010 *Mathematics Subject Classification.* 53A10, 53C42.

Key words and phrases. Biconservative surfaces, minimal surfaces, real space forms.

The second author's work was co-funded by the European Social Fund through Sectoral Operational Programme Human Resources Development 2007 – 2013, project number POSDRU/187/1.5/S/155397, project title "Towards a New Generation of Elite Researchers through Doctoral Scholarships".

The paper is organized as follows. After a short section where we recall some notions and results on biconservative submanifolds, we show, in the third section, that on a non-CMC biconservative surface $\left(M^2, g\right)$ in a space form $N^3(c)$ we can determine a new Riemannian metric g_r such that $\left(M^2, g_r\right)$ is a Ricci surface in $N^3(c)$. Then, in the last section of the paper, we obtain an intrinsic characterization of non-CMC biconservative surfaces in a space form.

Acknowledgments. The authors wish to thank Sergiu Moroianu for useful comments and discussions.

2. Preliminaries

As we have already seen, biharmonic maps $\psi : (M, g) \to (N, h)$, as suggested by J. Eells and J. H. Sampson [5], are the critical points of the bienergy functional. The corresponding Euler-Lagrange equation, obtained in [8], is

$$\tau_2(\psi) = -\Delta\tau(\psi) - \operatorname{trace} R^N(d\psi, \tau(\psi))d\psi = 0,$$

where $\tau_2(\psi)$ is the *bitension field* of ψ, $\Delta = -\operatorname{trace}(\nabla^\psi)^2 = -\operatorname{trace}(\nabla^\psi\nabla^\psi - \nabla^\psi_\nabla)$ is the rough Laplacian defined on sections of $\psi^{-1}(TN)$ and R^N is the curvature tensor of N, given by $R^N(X, Y)Z = [\bar\nabla_X, \bar\nabla_Y]Z - \bar\nabla_{[X,Y]}Z$.

The *stress-energy tensor* associated to a variational problem, described in [7] by D. Hilbert, is a symmetric 2-covariant tensor S conservative at critical points, i.e., S satisfies $\operatorname{div} S = 0$ at these points.

P. Baird and J. Eells [1] and A. Sanini [19] used such a tensor given by

$$S = \frac{1}{2}|d\psi|^2 g - \psi^* h$$

to study harmonic maps. It has been proved that S satisfies the equation

$$\operatorname{div} S = -\langle\tau(\psi), d\psi\rangle,$$

which implies that $\operatorname{div} S$ vanishes when ψ is harmonic. When $\psi : M \to N$ is an isometric immersion, $\tau(\psi)$ is normal and then $\operatorname{div} S = 0$ always holds in this case.

Consider now the stress-energy tensor S_2 of the bienergy. This tensor, that was studied for the first time in [9] and then in papers like [4, 6, 11, 13, 14], is given by

$$\begin{aligned} S_2(X, Y) =& \frac{1}{2}|\tau(\psi)|^2\langle X, Y\rangle + \langle d\psi, \nabla\tau(\psi)\rangle\langle X, Y\rangle \\ &- \langle d\psi(X), \nabla_Y\tau(\psi)\rangle - \langle d\psi(Y), \nabla_X\tau(\psi)\rangle \end{aligned}$$

and it satisfies

$$\operatorname{div} S_2 = -\langle\tau_2(\psi), d\psi\rangle.$$

If $\psi : M \to N$ is an isometric immersion, then we have $(\operatorname{div} S_2)^\sharp = -\tau_2(\psi)^\top$ and, therefore, $\operatorname{div} S_2$ does not automatically vanish.

Definition 2.1. A submanifold $\psi : M \to N$ of a Riemannian manifold N is called a *biconservative submanifold* if $\operatorname{div} S_2 = 0$, i.e., $\tau_2(\psi)^\top = 0$.

The biharmonic equation $\tau_2(\psi) = 0$ of a submanifold $\psi : M \to N$ can be decomposed in its normal and tangent part (see [3, 16]). In the case of hypersurfaces M in N, we get

$$\Delta f + f|A|^2 - f \operatorname{Ricci}^N(\eta, \eta) = 0$$

and

$$2A(\nabla f) + f\nabla f - 2f(\operatorname{Ricci}^N(\eta))^\top = 0,$$

where η is the unit normal of M in N, A is the shape operator, $f = \operatorname{trace} A$ is the mean curvature function, and $(\operatorname{Ricci}^N(\eta))^\top$ is the tangent component of the Ricci curvature of N in the direction of η.

From this decomposition, it follows that a surface $\psi : M^2 \to N^3(c)$ in a space form $N^3(c)$, i.e., a 3-dimensional simply connected complete manifold with constant sectional curvature c, is biconservative if and only if

$$A(\nabla f) = -\frac{f}{2}\nabla f. \tag{2.1}$$

It is then easy to see that any CMC surface in $N^3(c)$ is biconservative and, therefore, when studying biconservative surfaces in space forms we are interested in the non-CMC case. We should, however, mention that, in the general case, if M is a biconservative surface in an n-dimensional Riemannian manifold N, then it has constant mean curvature if and only if the $(2,0)$-part of the quadratic form Q, defined on M by $Q(X,Y) = \langle B(X,Y), H\rangle$, is holomorphic, where B is the second fundamental form of M in N and H is the mean curvature vector field (see [12, 14]).

We end this section recalling the following result on non-CMC biconservative surfaces in $N^3(c)$ that we will use later on.

Theorem 2.2 ([4])**.** *Let M^2 be a non-CMC biconservative surface in a space form $N^3(c)$. There exists an open subset $U \subset M$ such that, on U, the Gaussian curvature K of M satisfies*

$$K = \det A + c = -\frac{3f^2}{4} + c \tag{2.2}$$

and

$$(c-K)\Delta K - |\nabla K|^2 - \frac{8}{3}K(c-K)^2 = 0, \tag{2.3}$$

where A is the shape operator of M in N, $f = \operatorname{trace} A$ is the mean curvature function, and Δ is the Laplace-Beltrami operator on M.

Remark 2.3. It is easy to see, from (2.2), that the Gaussian curvature K of a non-CMC biconservative surface in $N^3(c)$ satisfies $c - K > 0$.

Convention. Henceforth, all surfaces are assumed to be connected and oriented.

3. Biconservativity and minimality in space forms

A Riemannian surface $\left(M^2, g\right)$ with Gaussian curvature K is said to satisfy the *Ricci condition* if $c - K > 0$ and the metric $(c-K)^{1/2}g$ is flat, where $c \in \mathbb{R}$ is a constant. In this case, $\left(M^2, g\right)$ is called a *Ricci surface*. G. Ricci-Curbastro [17] proved that, when $c = 0$, a surface satisfying the Ricci condition can be locally isometrically embedded in $\mathbb{R}^3$ as a minimal surface. Actually, there exists a one-parameter family of such embeddings. H. B. Lawson [10, Theorem 8] generalized this result by showing that the Ricci condition is an intrinsic characterization of minimal surfaces in space forms $N^3(c)$, with constant sectional curvature c (see also [18]).

In the following, we will see that the Ricci condition, as stated above, is equivalent to an equation that looks very much like equation (2.3), satisfied by the Gaussian curvature of a non-CMC biconservative surface in a space form $N^3(c)$. Then, a natural question is whether there exists a simple way to transform surfaces satisfying (2.3) in Ricci surfaces in $N^3(c)$. As it will turn out, the answer to this question is affirmative.

We will first briefly recall some known results in conformal geometry. Let (M^2, g) be a Riemannian surface with Gaussian curvature K and Laplacian Δ. Consider a new Riemannian metric $\bar{g} = e^{2\varphi} g$ on M, where $\varphi \in C^\infty(M)$. If $\bar{\Delta}$ and $\bar{K}$ are the Laplacian and the Gaussian curvature, respectively, of $\bar{g}$, then we have (see [2]):

$$\bar{\Delta} = e^{-2\varphi}\Delta \tag{3.1}$$

and

$$\bar{K} = e^{-2\varphi}(K + \Delta\varphi). \tag{3.2}$$

The following proposition points out some equivalent characterizations of Ricci surfaces.

Proposition 3.1. *Let $\left(M^2, g\right)$ be a Riemannian surface such that its Gaussian curvature K satisfies $c - K > 0$, where $c \in \mathbb{R}$ is a constant. Then, the following conditions are equivalent*:

(i) *K satisfies*

$$(c - K)\Delta K - |\nabla K|^2 - 4K(c - K)^2 = 0; \tag{3.3}$$

(ii) *K satisfies*

$$\Delta \log(c - K) + 4K = 0; \tag{3.4}$$

(iii) *the metric $(c - K)^{1/2} g$ is flat.*

Moreover, if $c = 0$, then we also have a fourth equivalent condition:

(iv) *the metric $(-K)g$ has constant Gaussian curvature equal to* 1.

Proof. First, we easily get that

$$\Delta \log(c - K) = \frac{(K - c)\Delta K + |\nabla K|^2}{(c - K)^2},$$

which implies that (i) and (ii) are equivalent.

Next, in the same way as in [15], we consider a family of Riemannian metrics on M given by $g_r = (c - K)^r g$, where $r \in \mathbb{R}$ is a constant. From equation (3.2), one obtains that the Gaussian curvature curvature K_r of g_r is given by

$$K_r = (c - K)^{-r}\left(K + \frac{1}{2}\Delta \log(c - K)^r\right).$$

If (ii) holds then $K_r = (1 - 2r)(c - K)^{-r} K$ and, therefore, (ii) implies (iii) and (iv). Conversely, it is easy to see, from the expression of K_r, that (iii) implies (ii) and also, if $c = 0$, (iv) implies (ii). $\square$

Remark 3.2. Proposition 3.1 was first proved in the case when $c = 0$ in [15].

Working exactly as in the proof of Proposition 3.1 we get our following result.

Proposition 3.3. *Let $\left(M^2, g\right)$ be a Riemannian surface such that its Gaussian curvature K satisfies $c - K > 0$, where $c \in \mathbb{R}$ is a constant. Then, the following conditions are equivalent*:

(i) *K satisfies equation* (2.3);
(ii) $\Delta \log(c - K) + (8/3)K = 0$;
(iii) *the metric $(c - K)^{3/4} g$ is flat.*

Moreover, if $c = 0$, then we also have a fourth equivalent condition:

(iv) *the metric $(-K)g$ has constant Gaussian curvature equal to* $1/3$.

Now, we can state our first main result.

Theorem 3.4. *Let $\left(M^2, g\right)$ be a Riemannian surface with negative Gaussian curvature K that satisfies*

$$K\Delta K + |\nabla K|^2 + \frac{8}{3}K^3 = 0. \tag{3.5}$$

Then $\left(M^2, (-K)^{1/2} g\right)$ is a Ricci surface in $\mathbb{R}^3$.

Proof. From Proposition 3.1, one can see that suffices to show that there exists a Riemannian metric on M, conformally equivalent to g, that satisfies (3.4).

In order to find such a metric, let us consider again the metrics $g_r = (-K)^r g$, with $r \in \mathbb{R}$. From (3.2) and Proposition 3.3, one obtains that the Gaussian curvature curvature K_r of g_r is given by

$$K_r = (-K)^{-r}\left(K + \frac{1}{2}\Delta \log(-K)^r\right) = -\frac{3-4r}{3}(-K)^{1-r}.$$

Assume that $3 - 4r > 0$, i.e., $K_r < 0$, and then, using equations (3.1) and (3.5) and Proposition 3.3, we can compute

$$\begin{aligned}
\Delta_r \log(-K_r) =& \Delta_r \log\left(\frac{3-4r}{3}(-K)^{1-r}\right) = (1-r)\Delta_r \log(-K)\\
=&(1-r)(-K)^{-r}\Delta \log(-K)\\
=&\frac{8(1-r)}{3}(-K)^{1-r},
\end{aligned}$$

where Δ_r is the Laplacian of g_r. Now, equation (3.4) becomes

$$0 = \Delta_r \log(-K_r) + 4K_r = \Delta_r \log(-K_r) - \frac{4(3-4r)}{3}(-K)^{1-r} = \frac{4(2r-1)}{3}(-K)^{1-r}$$

and we get that $r = 1/2$.

We have just proved that $\left(M^2, g_{1/2} = (-K)^{1/2} g\right)$ is a Ricci surface with Gaussian curvature $K_{1/2} = -(1/3)(-K)^{1/2} < 0$. $\square$

From Theorems 2.2 and 3.4, one obtains the following corollary.

Corollary 3.5. *Let $\left(M^2, g\right)$ be a biconservative surface in $\mathbb{R}^3$, where g is the induced metric on M. If $f(p) > 0$ and $(\nabla f)(p) \neq 0$ at any point $p \in M$, where f is the mean curvature function, then $\left(M^2, (-K)^{1/2} g\right)$ is a Ricci surface.*

Remark 3.6. In the same way as in Theorem 3.4 one can show that if $\left(M^2, g\right)$ is a Ricci surface in $\mathbb{R}^3$ with negative Gaussian curvature K, then the Gaussian curvature of $\left(M^2, (-K)^{-1} g\right)$ is negative and satisfies equation (3.5).

Although the method used to prove Theorem 3.4 does not work in the case of non-flat space forms, it is still possible to extend this result to the case of space forms, as shown by the following theorem.

Theorem 3.7. *Let $\left(M^2, g\right)$ be a biconservative surface in a space form $N^3(c)$ with induced metric g and Gaussian curvature K. If $f(p) > 0$ and $(\nabla f)(p) \neq 0$ at any point $p \in M$, where f is the mean curvature function, then, on an open dense set, $\left(M^2, (c-K)^r g\right)$ is a Ricci surface in $N^3(c)$, where r is a locally defined function that satisfies*

$$K + \Delta\left(\frac{1}{4}\log(c - K_r) + \frac{r}{2}\log(c - K)\right) = 0,$$

with the Gaussian curvature K_r of $(c-K)^r g$ given by

$$K_r = (c-K)^{-r}\left(\frac{3-4r}{3}K + \frac{1}{2}\log(c-K)\Delta r + (c-K)^{-1}g(\nabla r, \nabla K)\right).$$

Proof. Let A be the shape operator of M^2 in $N^3(c)$ and then $f = \operatorname{trace} A$ is the mean curvature function. Working as in [4], we define on M a global orthonormal frame field $\{X_1, X_2\}$, where $X_1 = (\nabla f)/|\nabla f|$.

In [4] it is proved that $X_2 f = 0$, which implies, using Theorem 2.2, that also $X_2 K = 0$. In the same paper it is shown that

$$AX_1 = -\frac{f}{2}X_1, \quad AX_2 = \frac{3f}{2}X_2$$

and

$$\nabla_{X_1}X_1 = \nabla_{X_1}X_2 = 0, \quad \nabla_{X_2}X_1 = -\frac{3X_1 f}{4f}X_2, \quad \nabla_{X_2}X_2 = \frac{3X_1 f}{4f}X_1, \tag{3.6}$$

where ∇ is the induced connection on M.

Now, let us consider a family of Riemannian metrics $g_r = (c-K)^r g$ on M, this time r being a function on M such that $X_2 r = 0$. From the above formulas for the Levi-Civita connection ∇, it easily follows that $[X_1, X_2](K) = 0$ and $[X_1, X_2](r) = 0$. Therefore, we also have $X_2(X_1 K) = 0$ and $X_2(X_1 r) = 0$.
From (3.2), we have that the Gaussian curvature K_r of g_r is given by

$$K_r = (c-K)^{-r}\left(K + \frac{1}{2}\Delta(r\log(c-K))\right),$$

where K is the Gaussian curvature of g, and, since, after a straightforward computation, also using Theorem 2.2 and Proposition 3.3, we have that

$$\begin{aligned}\Delta(r\log(c-K)) =& \sum_{i=1}^{2}\{-X_i(X_i(r\log(c-K))) + \nabla_{X_i}X_i(r\log(c-K))\}\\ =& r\Delta\log(c-K) + \log(c-K)\Delta r - 2g(\nabla r, \nabla\log(c-K))\\ =& -\frac{8}{3}rK + \log(c-K)\Delta r + 2(c-K)^{-1}g(\nabla r, \nabla K),\end{aligned} \tag{3.7}$$

it follows that

$$K_r = (c-K)^{-r}\left(\frac{3-4r}{3}K + \frac{1}{2}\log(c-K)\Delta r + (c-K)^{-1}g(\nabla r, \nabla K)\right). \tag{3.8}$$

Next, assume that $(c-K_r)(p) > 0$ at any point $p \in M$ and consider a new Riemannian metric $\bar{g}$ on M given by

$$\begin{aligned}\bar{g} =& (c-K_r)^{1/2}g_r = (c-K_r)^{1/2}(c-K)^r g\\ =& e^{2\varphi}g.\end{aligned}$$

We ask the corresponding Gaussian curvature $\bar{K}$ to vanish. From the definition of $\bar{g}$, one obtains

$$\varphi = \frac{1}{4}\log(c-K_r) + \frac{r}{2}\log(c-K). \tag{3.9}$$

Equation (3.2) implies that

$$\bar{K} = (c-K_r)^{-1/2}(c-K)^{-r}(K + \Delta\varphi)$$

and then $\bar{K} = 0$ becomes

$$K + \Delta\varphi = 0. \tag{3.10}$$

Using (3.9), (3.7), and (3.8) we get

$$\begin{aligned}
\Delta\varphi =&\frac{1}{4}\Delta\log(c-K_r)+\frac{1}{2}\Delta(r\log(c-K))\\
=&\frac{1}{4}(c-K_r)^{-2}((K_r-c)\Delta K_r+|\nabla K_r|^2)-\frac{4}{3}rK+\frac{1}{2}\log(c-K)\Delta r\\
&+(c-K)^{-1}g(\nabla r,\nabla K)\\
=&\frac{1}{4}(c-K_r)^{-2}\Big\{(K_r-c)\Delta\Big((c-K)^{-r}\Big(\frac{3-4r}{3}K+\frac{1}{2}\log(c-K)\Delta r\\
&+(c-K)^{-1}g(\nabla r,\nabla K)\Big)\Big)\\
&+\Big|\nabla\Big((c-K)^{-r}\Big(\frac{3-4r}{3}K+\frac{1}{2}\log(c-K)\Delta r+(c-K)^{-1}g(\nabla r,\nabla K)\Big)\Big)\Big|^2\Big\}\\
&-\frac{4}{3}rK+\frac{1}{2}\log(c-K)\Delta r+(c-K)^{-1}g(\nabla r,\nabla K).
\end{aligned}$$

We note that, since $\nabla K\neq 0$ at any point, the function $\log(c-K)$ cannot vanish on an open subset of M. Now, away from the points where $\log(c-K)=0$, using the above equation, (3.6), and (2.2), equation (3.10) can be written as

$$\Delta^2 r=F(r,X_1r,X_1(X_1r),X_1(X_1(X_1r))), \tag{3.11}$$

where the coefficients in the expression F in the right hand side are smooth functions on M depending on K, $X_1(K)$, $X_1(X_1K)$, and $X_1(X_1(X_1K))$.

Let us consider a point $p_0\in M$ and $\gamma=\gamma(u)$ an integral curve of X_1 with $\gamma(0)=p_0$. Let ϕ be the flow of X_2 and, in a neighborhood $U\subset M$ of p_0, define a local parametrization of M,

$$X(u,s)=\phi_{\gamma(u)}(s)=\phi(\gamma(u),s).$$

We have $X(u,0)=\phi_{\gamma(u)}(0)=\gamma(u)$,

$$X_u(u,0)=\partial_u(u,0)=\gamma'(u)=X_1(\gamma(u))=X_1(u,0),$$

and

$$X_s(u,s)=\partial_s(u,s)=\phi'_{\gamma(u)}(s)=X_2(\phi_{\gamma(u)}(s))=X_2(u,s),$$

for any u and s.

By hypothesis, we have $X_2r=0$, which means that $r=r(u)$ on U. Moreover, since $X_2(X_1r)=0$, it follows that $(X_1r)(u,s)=(X_1r)(u,0)=r'(u)$ on U. From the formulas of the Levi-Civita connection, it is easy to see that also $X_2(X_1(X_1r))=0$, $X_2(X_1(X_1(X_1r)))=0$, and $X_2(X_1(X_1(X_1(X_1r))))=0$, that implies $X_1(X_1r)=r''(u)$, $X_1(X_1(X_1r))=r'''(u)$, and $X_1(X_1(X_1(X_1r)))=r^{(iv)}(u)$, respectively. Moreover, the same formulas hold if we take K instead of r. Therefore, on U, equation (3.11) becomes

$$r^{(iv)}(u)=\tilde F(u,r(u),r'(u),r''(u),r'''(u)). \tag{3.12}$$

The initial conditions follow from $(c-K_r)(p_0)>0$, i.e., from

$$\begin{aligned}
(c-K(0))^{-r(0)}\Big(\frac{3-4r(0)}{3}K(0)+\frac{1}{2}\log(c-K(0))\Big(-r''(0)+\frac{3f'(0)}{4f(0)}r'(0)\Big)&\\
+(c-K(0))^{-1}r'(0)K'(0)\Big)&<c.
\end{aligned}$$

It is easy to see that we can choose $r(0)$, $r'(0)$, and $r''(0)$ such that the above inequality is satisfied.

Now, from the ODE's theory, we know that equation (3.12), with the given initial conditions, has a unique solution, which means that there exists a flat Riemannian metric $\bar{g} = (c - K_r)^{1/2} g_r$ on U, and then we use [10, Theorem 8] to conclude that our surface (M^2, g) can be locally conformally embedded in $N^3(c)$ as a minimal surface. $\square$

Remark 3.8. It is straightforward to verify that, when $c = 0$, the only constant solution of equation (3.11) is $r = 1/2$.

4. An intrinsic characterization of biconservative surfaces in space forms

While any of the equivalent conditions in Proposition 3.1 characterizes intrinsically minimal surfaces in 3-dimensional space forms $N^3(c)$ (see [10, Theorem 8]), the similar conditions in Proposition 3.3 alone fail to do the same in the case of biconservative surfaces. In this section, we will find the intrinsic necessary and sufficient conditions for a Riemannian surface to be locally embedded in $N^3(c)$ as a non-CMC biconservative surface.

We will first need the following theorem.

Theorem 4.1. *Let $\left(M^2, g\right)$ be a Riemannian surface with Gaussian curvature K satisfying $(\nabla K)(p) \neq 0$ and $c - K(p) > 0$ at any point $p \in M$, where $c \in \mathbb{R}$ is a constant. Let $X_1 = (\nabla K)/|\nabla K|$ and $X_2 \in C(TM)$ be two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$. If level curves of K are circles in M with constant curvature*

$$\kappa = \frac{3X_1 K}{8(c-K)} = \frac{3|\nabla K|}{8(c-K)},$$

then, for any point $p_0 \in M$, there exists a parametrization $X = X(u,s)$ of M in a neighborhood $U \subset M$ of p_0 positively oriented such that

(a) *the curve $u \to X(u,0)$ is an integral curve of X_1 with $X(0,0) = p_0$ and $s \to X(u,s)$ is an integral curve of X_2, for any u;*
(b) *$K(u,s) = (K \circ X)(u,s) = (K \circ X)(u,0) = K(u)$, for any (u,s);*
(c) *for any pair (u,s), we have*

$$g_{11}(u,s) = \frac{9}{64}\left(\frac{K'(u)}{c-K(u)}\right)^2 s^2 + 1, \quad g_{12}(u,s) = -\frac{3K'(u)}{8(c-K(u))}s, \quad g_{22}(u,s) = 1;$$

(d) *the Gaussian curvature $K = K(u)$ satisfies*

$$24(c-K)K'' + 33(K')^2 + 64K(c-K)^2 = 0;$$

(e) *$X_1 = X_u - g_{12}X_s$, $X_2 = X_s$, the Levi-Civita connection on $\left(M^2, g\right)$ is given by*

$$\nabla_{X_1} X_1 = \nabla_{X_1} X_2 = 0, \quad \nabla_{X_2} X_2 = -\frac{3X_1 K}{8(c-K)} X_1, \quad \nabla_{X_2} X_1 = \frac{3X_1 K}{8(c-K)} X_2,$$

and, therefore, the integral curves of X_1 are geodesics.

Proof. Let p_0 be a fixed point in M, $\gamma = \gamma(u)$ an integral curve of X_1 with $\gamma(0) = p_0$, and ϕ the flow of X_2. Consider again

$$X(u,s) = \phi_{\gamma(u)}(s) = \phi(\gamma(u), s).$$

As we have already seen, we have $X(u,0) = \phi_{\gamma(u)}(0) = \gamma(u)$,

$$X_u(u,0) = \partial_u(u,0) = \gamma'(u) = X_1(\gamma(u)) = X_1(u,0),$$

and

$$X_s(u,s) = \partial_s(u,s) = \phi'_{\gamma(u)}(s) = X_2(\phi_{\gamma(u)}(s)) = X_2(u,s),$$

for any u and s.

Since $X_s(u,s) = X_2(u,s)$, it follows that $|X_s(u,s)|^2 = 1$, which means that

$$g_{22}(u,s) = 1. \tag{4.1}$$

We also have, for any u,

$$g_{11}(u,0) = |X_u(u,0)|^2 = 1, \quad g_{12}(u,0) = g(X_u(u,0), X_s(u,0)) = 0. \tag{4.2}$$

We will now find the expression of X_1 with respect to $X_u = \partial_u$ and $X_s = \partial_s$. We write $X_1 = a\partial_u + b\partial_s$, where a and b are smooth functions. Using (4.1), it follows that

$$1 = g(X_1, X_1) = a^2 g_{11} + 2ab g_{12} + b^2 g_{22} = a^2 g_{11} + 2ab g_{12} + b^2,$$

and

$$0 = g(X_1, X_2) = a g_{12} + b g_{22} = a g_{12} + b.$$

From the second equation, one obtains $b = -a g_{12}$ and, replacing in the first one, we get $1 = a^2(g_{11} - g_{12}^2)$. Let us denote $\sigma(u,s) = \sqrt{g_{11} - g_{12}^2} > 0$ and then we have

$$X_1 = \frac{1}{\sigma}\partial_u - \frac{g_{12}}{\sigma}\partial_s. \tag{4.3}$$

Next, we note that, from the definition of X_1 and X_2, one obtains $X_2 K = 0$, i.e., the integral curves $s \to \phi_{\gamma(u)}(s)$ of X_2 are the level curves of K, that means that $s \to K(\phi_{\gamma(u)}(s))$ is a constant function. Also, identifying K with $K \circ X$, we can write $K = K(u,s)$. Since $X_2 K = 0$, it follows that actually $K(u,s) = K(u,0) = K(u)$, for any pair (u,s). The level curves $s \to \phi_{\gamma(u)}(s)$ of K are parametrized by arc length and, by hypothesis, are circles with constant curvature $\kappa = 3X_1K/(8(c-K))$, which means, also using (4.3) and the fact that $\kappa(u,s) = \kappa(u)$, that

$$\begin{aligned} X_1 K =& \frac{8}{3}\kappa(c-K) = \frac{8}{3}\kappa(u)(c - K(u)) \\ =& \frac{1}{\sigma}K' = \frac{1}{\sigma(u,s)}K'(u), \end{aligned} \tag{4.4}$$

which implies that $X_2(X_1 K) = 0$ and $\sigma(u,s) = \sigma(u) = 1$, for any u and s.

Let us consider a fixed u. As $\{X_2, -X_1\}$ is positively oriented, we have

$$\begin{aligned} \nabla_{\phi'_{\gamma(u)}(s)}\phi'_{\gamma(u)}(s) =& \nabla_{X_2}X_2 = \kappa(-X_1) \\ =& \Gamma_{22}^1 \partial_u + \Gamma_{22}^2 \partial_s \end{aligned}$$

and then

$$\begin{aligned} \kappa =& g\left(\nabla_{\phi'_{\gamma(u)}(s)}\phi'_{\gamma(u)}(s), -X_1\right) = g\left(\Gamma_{22}^1\partial_u + \Gamma_{22}^2\partial_s, -\partial_u + g_{12}\partial_s\right) \\ =& -\Gamma_{22}^1\left(g_{11} - g_{12}^2\right) = -\Gamma_{22}^1, \end{aligned}$$

where Γ_{ij}^k are the Christoffel symbols.

Since, by the definition of Γ_{22}^1, using $1 = \sigma^2 = g_{11} - g_{12}^2$, we have

$$\begin{aligned} \Gamma_{22}^1 =& \frac{1}{2}g^{11}\left(\frac{\partial g_{21}}{\partial s} + \frac{\partial g_{21}}{\partial s} - \frac{\partial g_{22}}{\partial u}\right) + \frac{1}{2}g^{12}\left(\frac{\partial g_{22}}{\partial s} + \frac{\partial g_{22}}{\partial s} - \frac{\partial g_{22}}{\partial s}\right) \\ =& g^{11}\frac{\partial g_{12}}{\partial s} = \frac{\partial g_{12}}{\partial s}, \end{aligned}$$

one obtains $\kappa = -\partial g_{12}/\partial s$. From equation (4.4), it follows that

$$K'(u) = -\frac{8}{3}\frac{\partial g_{12}}{\partial s}(c - K(u)),$$

which leads to

$$\frac{\partial g_{12}}{\partial s} = -\frac{3K'(u)}{8(c-K(u))} = \frac{3}{8}(\log(c - K(u)))'$$

and, therefore,

$$g_{12}(u,s) = -\frac{3K'(u)}{8(c-K(u))}s + \alpha(u).$$

But, from (4.2), we know that $g_{12}(u,0) = 0$, which implies that $\alpha(u) = 0$, and we conclude that

$$g_{12}(u,s) = -\frac{3K'(u)}{8(c-K(u))}s. \tag{4.5}$$

Finally, since $1 = \sigma^2 = g_{11} - g_{12}^2$, we find

$$g_{11}(u,s) = \frac{9}{64}\left(\frac{K'(u)}{c-K(u)}\right)^2 s^2 + 1. \tag{4.6}$$

Now, from the definition of Christoffel symbols and (4.1), (4.5), and (4.6), we obtain, after a straightforward computation, that the Gauss equation of (M^2, g)

$$K = -\frac{1}{g_{11}}\left\{(\Gamma_{12}^2)_u - (\Gamma_{11}^2)_s + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2\right\}$$

is equivalent to

$$24(c-K)K'' + 33(K')^2 + 64K(c-K)^2 = 0. \tag{4.7}$$

Again using the definition of Christoffel symbols and equations (4.1), (4.5), (4.6), the expressions of X_1 and X_2, and equation (4.7), we get the formulas for the Levi-Civita connection as given by the last item of the theorem. □

Remark 4.2. It is easy to verify that, in the hypotheses of Theorem 4.1, equation

$$24(c-K)K'' + 33(K')^2 + 64K(c-K)^2 = 0$$

can be written as

$$(c-K)\Delta K - |\nabla K|^2 - \frac{8}{3}K(c-K)^2 = 0.$$

Remark 4.3. Considering a change of coordinates $(u,s) \to \left(u, (c-K)^{3/8}s\right) = (u,v)$ in Theorem 4.1, we obtain, after a straightforward computation, a simpler expression

$$g = du^2 + (c-K)^{-\frac{3}{4}}dv^2$$

for the Riemannian metric on the surface. Moreover, if we consider a second change of coordinates $(u,v) \to \left(\int_{u_0}^{u}(c-K)^{3/8}du, v\right) = (\tilde{u}, \tilde{v})$, then the metric g can be written as

$$g = (c-K)^{-\frac{3}{4}}\left(d\tilde{u}^2 + d\tilde{v}^2\right),$$

where $K(\tilde{u}) = K(u(\tilde{u}))$, which means that $(\tilde{u}, \tilde{v})$ are isothermal coordinates on the surface.

The converse of Theorem 4.1 is the following result, that can be proved by a straightforward computation.

Theorem 4.4. *Let M^2 be a surface and $c \in \mathbb{R}$ a constant. Consider a fixed point $p_0 \in M$, a parametrization $X = X(u,s)$ of M on a neighborhood $U \subset M$ of p_0 positively oriented, and $K = K(u)$ a function on M such that $K'(u) > 0$ and $c - K(u) > 0$, for any u, and*

$$24(c-K)K'' + 33(K')^2 + 64K(c-K)^2 = 0.$$

Define a Riemannian metric $g = g_{11}du^2 + 2g_{12}duds + g_{22}ds^2$ on U by

$$g_{11}(u,s) = \frac{9}{64}\left(\frac{K'(u)}{c-K(u)}\right)^2 s^2 + 1, \quad g_{12}(u,s) = -\frac{3K'(u)}{8(c-K(u))}s, \quad g_{22}(u,s) = 1.$$

Then K is the Gaussian curvature of g and its level curves, i.e., the curves $s \to X(u,s)$, are circles in M with curvature $\kappa = 3K'(u)/(8(c-K(u)))$.

We are now ready to prove the main result of this section, which provides an intrinsic characterization of non-CMC biconservative surfaces in a 3-dimensional space form $N^3(c)$.

Theorem 4.5. *Let (M^2, g) be a Riemannian surface and $c \in \mathbb{R}$ a constant. Then M can be locally isometrically embedded in a space form $N^3(c)$ as a biconservative surface with positive mean curvature having the gradient different from zero at any point $p \in M$ if and only if the Gaussian curvature K satisfies $c - K(p) > 0$, $(\nabla K)(p) \neq 0$, and its level curves are circles in M with curvature $\kappa = (3|\nabla K|)/(8(c-K))$.*

Proof. The direct implication was proved in [4] and [6].

To prove the converse, let us consider $X_1 = (\nabla K)/|\nabla K|$ and $X_2 \in C(TM)$ two vector fields such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$. From Theorem 4.1 we have seen that the Levi-Civita connection on (M^2, g) is given by

$$\nabla_{X_1}X_1 = \nabla_{X_1}X_2 = 0, \quad \nabla_{X_2}X_2 = -\frac{3X_1K}{8(c-K)}X_1, \quad \nabla_{X_2}X_1 = \frac{3X_1K}{8(c-K)}X_2.$$

Now, consider $f = (2/\sqrt{3})\sqrt{c-K} > 0$ and, since $X_2K = 0$, we easily get

$$\nabla f = -\frac{X_1K}{\sqrt{3(c-K)}}X_1 = -\frac{\nabla K}{\sqrt{3(c-K)}}.$$

Define $\tilde{X}_1 = (\nabla f)/|\nabla f| = -X_1$ and $\tilde{X}_2 = -X_2$ and then

$$\nabla_{\tilde{X}_1}\tilde{X}_1 = \nabla_{\tilde{X}_1}\tilde{X}_2 = 0$$

and

$$\nabla_{\tilde{X}_2}\tilde{X}_2 = \nabla_{X_2}X_2 = -\frac{3\tilde{X}_1K}{8(c-K)}\tilde{X}_1, \quad \nabla_{\tilde{X}_2}\tilde{X}_1 = \nabla_{X_2}X_1 = \frac{3\tilde{X}_1K}{8(c-K)}\tilde{X}_2.$$

Since $(\tilde{X}_1 f)/f = -(\tilde{X}_1K)/(2(c-K))$, we obtain

$$\nabla_{\tilde{X}_2}\tilde{X}_2 = \frac{3\tilde{X}_1 f}{4f}\tilde{X}_1 \quad \text{and} \quad \nabla_{\tilde{X}_2}\tilde{X}_1 = -\frac{3\tilde{X}_1 f}{4f}\tilde{X}_2.$$

Let us now consider a tensor field A of type $(1,1)$ on M defined by

$$A\tilde{X}_1 = -\frac{f}{2}\tilde{X}_1 \quad \text{and} \quad A\tilde{X}_2 = \frac{3f}{2}\tilde{X}_2.$$

It is straightforward to verify that A satisfies the Gauss equation

$$K = c + \det A$$

and the Codazzi equation

$$(\nabla_{\tilde{X}_1} A)\tilde{X}_2 = (\nabla_{\tilde{X}_2} A)\tilde{X}_1,$$

which means that M can be locally isometrically embedded in $N^3(c)$ with A its shape operator. Moreover, from the definition of A it is easy to see that

$$A(\nabla f) = -\frac{f}{2}\nabla f,$$

and, from (2.1), it follows that M is a biconservative surface in $N^3(c)$. □

Remark 4.6. If the surface M in Theorem 4.5 is simply connected, then the theorem holds globally, but, in this case, instead of a local isometric embedding we have a global isometric immersion.

Remark 4.7. Let (M^2, g) be a simply connected Riemannian surface and $c \in \mathbb{R}$ a constant. If M admits two biconservative isometric immersions in $N^3(c)$ such that their mean curvatures are positive with gradients different from zero at any point $p \in M$, then the two immersions differ by an isometry of $N^3(c)$.

References

[1] P. Baird, J. Eells, *A conservation law for harmonic maps*, Geometry Symposium Utrecht 1980, 1–25, Lecture Notes in Math. 894, Springer, Berlin-New York, 1981.

[2] A. Besse, *Eistein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10, Springer-Verlag, Berlin, 1987.

[3] A. Balmuş, S. Montaldo, and C. Oniciuc, *Biharmonic PNMC submanifolds in spheres*, Ark. Mat. 51 (2013), 197–221.

[4] R. Caddeo, S. Montaldo, C. Oniciuc, and P. Piu, *Surfaces in three-dimensional space forms with divergence-free stress-bienergy tensor*, Ann. Mat. Pura Appl. (4) 193 (2014), 529–550.

[5] J. Eells, J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 (1964), 109–160.

[6] Y. Fu, *Explicit classification of biconservative surfaces in Lorentz* 3*-space forms*, Ann. Mat. Pura Appl., to appear.

[7] D. Hilbert, *Die grundlagen der physik*, Math. Ann. 92 (1924), 1–32.

[8] G. Y. Jiang, 2*-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A7(4) (1986), 389–402.

[9] G. Y. Jiang, *The conservation law for* 2*-harmonic maps between Riemannian manifolds*, Acta Math. Sinica 30 (1987), 220–225.

[10] H. B. Lawson, *Complete minimal surfaces in* $\mathbb{S}^3$, Ann. of Math. (2) 92 (1970), 335–374.

[11] E. Loubeau, S. Montaldo, and C. Oniciuc, *The stress-energy tensor for biharmonic maps*, Math. Z. 259 (2008), 503–524.

[12] E. Loubeau, C. Oniciuc, *Biharmonic surfaces of constant mean curvature*, Pacific J. Math. 271 (2014), 213–230.

[13] S. Montaldo, C. Oniciuc, and A. Ratto, *Proper biconservative immersions into the Euclidean space*, Ann. Mat. Pura Appl., to appear.

[14] S. Montaldo, C. Oniciuc, and A. Ratto, *Biconservative surfaces*, J. Geom. Anal., to appear.

[15] A. Moroianu, S. Moroianu, *Ricci surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci., to appear.

[16] Y.-L. Ou, *Biharmonic hypersurfaces in Riemannian manifolds*, Pacific J. Math. 248 (2010), 217–232.

[17] G. Ricci-Curbastro, *Sulla teoria intrinseca delle superficie ed in ispecie di quelle di* 2° *grado*, Ven. Ist. Atti (7) VI (1895), 445–488.

[18] M. Sakaki, *Minimal surfaces with the Ricci condition in* 4*-dimensional space forms*, Proc. Amer. Math. Soc. 121 (1994), 573–577.

[19] A. Sanini, *Applicazioni tra varietà riemanniane con energia critica rispetto a deformazioni di metriche*, Rend. Mat. 3 (1983), 53–63.

Department of Mathematics and Informatics, Gh. Asachi Technical University of Iasi, Bd. Carol I, 11, 700506 Iasi, Romania

E-mail address: dfetcu@math.tuiasi.ro

Faculty of Mathematics, Al. I. Cuza University of Iasi, Bd. Carol I, 11, 700506 Iasi, Romania

E-mail address: simona.nistor@math.uaic.ro

Faculty of Mathematics, Al. I. Cuza University of Iasi, Bd. Carol I, 11, 700506 Iasi, Romania

E-mail address: oniciucc@uaic.ro