CODERIVATIVE NECESSARY OPTIMALITY CONDITIONS FOR SHARP AND ROBUST EFFICIENCIES IN VECTOR OPTIMIZATION WITH VARIABLE ORDERING STRUCTURE

by

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Abstract: In this paper we define two new concepts of efficiency for vector optimization with variable ordering structure, namely the sharp and robust efficiencies, and we study their connections with classical concepts of efficiency in vector optimization. Then we get necessary optimality conditions for them by using Fréchet and Mordukhovich calculus coupled with the Gerstewitz's (Tammer's) scalarizing functional and openness results for set-valued maps.

Keywords: vector optimization · variable ordering structure · coderivative calculus · necessary optimality conditions

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1 Introduction

In the last years, the vector optimization field received a new impetus with the introduction of variable ordering structures. The main reference which gives a good "big picture" on this subject is [8]. Many efficiency notions are introduced and studied as well in [2, 9, 11]. In these mentioned references, the variable ordering structure is seen as a set-valued map from the output space of the objective mapping to itself.

A different perspective is proposed in [6], where the ordering mapping acts between the same spaces as the objective mapping. This was motivated by the authors of [6] that it is a natural requirement for both objective and ordering mappings to act between the same spaces. In this paper we follow the same approach and the objective in this context is twofold. On one hand, we introduce two new notions of minimum and we study their relations with the other notions and the classical efficiency concepts in vector optimization. On the other hand, we propose necessary optimality conditions in terms of Fréchet and Mordukhovich coderivatives of both objective and ordering mappings. The investigation for the second mentioned objective allows us to motivate the presence of the two notions of efficiency we study, since these are united by the methods we use. In both cases we work with the sum of the coderivatives of objective and ordering mappings.

In this paper we consider the so-called vector approach in set optimization which is different to the set approach, where the sets are compared directly by set relations (see [10] and the discussions and the references therein). The paper is organized as follows. In the second section we describe the two efficiency notions we deal with and we discuss their links with other notions in this field.

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The first concept that we define is the one of sharp efficiency for set-valued maps. The second one is a notion of robustness, because it allows to deal with different scenarios and to look for the best solutions that work well in all possible scenarios. Thus, the introduction of this concept is motivated by the fact that at the moment when we want to solve uncertain optimization problems, usually it is not known which scenario will occur (for an overview about robust optimization see [3, 14]). We compare these two concepts with several notions from literature. In Section 3 we introduce the main facts from generalized differentiation calculus we need in the sequel. Sections 4 and 5 propose optimality conditions for the two notions introduced in Section 2. The methods of study are different, but the final conclusions are comparable in a sense we make explicit in the last section.

2 Two new concepts of efficiency

Throughout this paper, we assume that X and Y are Banach spaces over the real field $\mathbb{R}$, unless otherwise stated. By $B(x,\varepsilon)$ we denote the open ball with center x and radius $\varepsilon > 0$ and by B_X the open unit ball of X. In the same manner, $D(x,\varepsilon)$ and D_X denote the closed balls. The symbol $\mathcal{V}(x)$ stands for the family of neighborhoods of x. We write $\operatorname{cl} A$ to denote the topological closure of a set A and we put $\operatorname{int} A$ for the topological interior of A. The notation X^* designates the topological dual of X, and by w^* we mean the weak star topology on X^*. If $K \subset Y$ is a cone, then its positive dual cone is defined by

$$K^* := \{y^* \in Y^* \mid y^*(y) \geq 0, \ \forall y \in K\}.$$

We denote the distance function $d_M : Y \to \mathbb{R}$ from $y \in Y$ to a nonempty set $M \subset Y$ by

$$d_M(y) \overset{not.}{=} d(y, M) := \inf_{m \in M} \|y - m\| \text{ for all } y \in Y.$$

Let $F : X \rightrightarrows Y$ be a set-valued map and $f : X \to \overline{\mathbb{R}}$ be a function. In the sequel, the symbols $\operatorname{Gr} F$, $\operatorname{dom} f$ and $\operatorname{epi} f$ denote the graph of F, the domain and the epigraph of f, respectively:

$$\begin{aligned}
\operatorname{Gr} F &:= \{(x,y) \in X \times Y \mid y \in F(x)\},\\
\operatorname{dom} f &:= \{x \in X \mid f(x) < \infty\},\\
\operatorname{epi} f &:= \{(x,t) \in X \times \mathbb{R} \mid f(x) \leq t\}.
\end{aligned}$$

Let $K : X \rightrightarrows Y$ be a set-valued map such that $K(x)$ is a closed, convex, proper and pointed cone in Y for any $x \in X$. This leads us, for every $x \in X$, to an order relation on $Y : y_1 \leq_{K(x)} y_2 \Leftrightarrow y_2 - y_1 \in K(x)$. Here the fact that $K(x)$ is proper means that $K(x) \neq \{0\}$ and $K(x) \neq Y$.

We consider the following optimization problem

$$(P_S) \text{ minimize } F(x), \text{ subject to } x \in S,$$

where S is a nonempty subset of X. If $S = X$, we denote the associated (unconstrained) problem by (P).

In our knowledge, no notion of sharp efficiency for set-valued mappings exists in the literature, where the efficiency is taken with respect to an order given by a set-valued map. We fill this gap with the following notion, which in the classical case was investigated under several versions in [12] and [5].

Definition 2.1 *Let $\varepsilon > 0$ and $\psi : (-\varepsilon, +\infty) \to \mathbb{R}$ be a nondecreasing function on $[0, +\infty)$ with the property that $\psi(t) = 0$ if and only if $t = 0$. One says that a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (S \times Y)$ is a weak ψ-sharp local nondominated point for (P_S) if there exist $\alpha > 0$ and a neighborhood U of $\overline{x}$ such that for every $x \in U \cap S$, $y \in F(x)$ one has*

$$d(y - \overline{y}, -K(x)) \geq \alpha\psi(d(x, W)), \tag{2.1}$$

where $W := \{x \in S \mid \overline{y} \in F(x)\}$.

This notion is a vectorial version of the notion of weak sharp minima in scalar optimization, in the sense made precise, for instance, in [20, Section 3.10]. The term "weak" refers here at the fact that the set of argmin points (that is, W) may be not a singleton. In the case where $W = \{\overline{x}\}$, we drop the word "weak" and then we have the notion of ψ-sharp local nondominated point. The notion of weak sharp minima is known to be an important tool in the sensitivity analysis of optimization problems, as well as in the convergence analysis of some optimization algorithms. However, in this paper our motivation is mainly theoretical, as further applications could be possible done later, on this basis.

The presence of ε in the above definitions will be needed for obtaining optimality conditions in terms of differential calculus, since we need the Fréchet differentiability at 0 for certain function (see Section 4). Notice that in general one can consider $\psi : [0, +\infty) \to \mathbb{R}$, but our small extension of the domain is not restrictive for the functions ψ usually considered: $\psi_1 : (-1, +\infty) \to \mathbb{R}$, $\psi_1(t) = \ln(1 + t)$, $\psi_2 : (-1, +\infty) \to \mathbb{R}$, $\psi_2(t) = \frac{1}{1+t}$, $\psi_3 : \mathbb{R} \to \mathbb{R}$, $\psi_3(t) = t$, $\psi_4 : \mathbb{R} \to \mathbb{R}$, $\psi_4(t) = \arctan t$.

Remark 2.2 *(i) Observe that, in case that $W = \{\overline{x}\}$ and $K(x) := Q$ for every $x \in (U \cap S) \setminus \{\overline{x}\}$, where Q is a proper closed convex cone in Y, the notion from the previous definition reduces to the notion of ψ-strict local minimizer for F over S, which was defined in [12].*

(ii) It is simple to observe that if $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a weak ψ-sharp local nondominated point for (P_S), then for the same α and U as in Definition 2.1, one has

$$d(y + z - \overline{y}, -K(x)) \geq \alpha\psi(d(x, W)),$$

for every $x \in U \cap S$, $y \in F(x)$ and $z \in K(x)$.

Indeed, from the above definition, relation (2.1) holds for every $x \in U \cap S$, $y \in F(x)$. Fix $x \in U \cap S$ and take $y \in F(x)$ and $z \in K(x)$. As $-K(x) - z \subset -K(x)$, we obtain that

$$d(y + z - \overline{y}, -K(x)) = d(y - \overline{y}, -K(x) - z) \geq d(y - \overline{y}, -K(x)),$$

and the thesis is proved. Obviously, this remark is true also for $S = X$.

Before introducing our second efficiency notion, we recall the concept of local (weakly) nondominated point from [6].

Definition 2.3 *Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$.*

(i) One says that $(\overline{x}, \overline{y})$ is a local nondominated point for F with respect to K if there is a neighborhood U of $\overline{x}$ such that, for every $x \in U$,

$$(F(x) - \overline{y}) \cap (-K(x)) \subset \{0\}.$$

(ii) If $\operatorname{int} K(x) \neq \emptyset$ *for every* x *in a neighborhood* V *of* $\overline{x}$*, then one says that* $(\overline{x}, \overline{y})$ *is a local weakly nondominated point for* F *with respect to* K *if there is a neighborhood* $U \subset V$ *of* $\overline{x}$ *such that, for every* $x \in U$*,*

$$(F(x) - \overline{y}) \cap (-\operatorname{int} K(x)) = \emptyset.$$

In this notion the argument x of the objective and ordering set-valued maps is the same, which means that the decision maker takes into account only the preferences in the moment of the decision. However, in the real world we often meet optimization problems with uncertain data. Thus, in the following we define the second new notion of minimum we deal with in this paper, which is in fact a robust solution, since it is immunized against the effect of data uncertainty.

Definition 2.4 *Let* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$*.*

(i) One says that $(\overline{x}, \overline{y})$ *is a local robust efficient point for* F *with respect to* K *if there is a neighborhood* U *of* $\overline{x}$ *such that, for every* x*,* $z \in U$*,*

$$(F(x) - \overline{y}) \cap (-K(z)) \subset \{0\}.$$

(ii) If $\operatorname{int} K(z) \neq \emptyset$ *for every* z *in a neighborhood* V *of* $\overline{x}$*, then one says that* $(\overline{x}, \overline{y})$ *is a local robust weakly efficient point for* F *with respect to* K *if there is a neighborhood* $U \subset V$ *of* $\overline{x}$ *such that, for every* x*,* $z \in U$*,*

$$(F(x) - \overline{y}) \cap (-\operatorname{int} K(z)) = \emptyset.$$

Remark 2.5 *(i) Observe that, if* $K(x) := Q$ *for every* $x \in U$*, where by* Q *we have denoted a closed, convex, proper and pointed cone in* Y*, then the notions given by the previous definition reduce to the classical ones: the Pareto minimality and weak Pareto minimality of* F *with respect to* Q*, respectively.*

(ii) In the previous definition, if we take $z = x$ *for any* $x \in U$*, then we get the notion of local (weakly) nondominated point for* F *with respect to* K*. Thus we have the following implication: if* $(\overline{x}, \overline{y})$ *is a local robust (weakly) efficient point for* F *with respect to* K*, then* $(\overline{x}, \overline{y})$ *is a local (weakly) nondominated point for* F *with respect to* K*. However, the reverse implication is not true. In this sense, we give the following example.*

Example 2.6 Let $X = [-1, 1]$, $Y = \mathbb{R}^2$, F, $K : [-1, 1] \rightrightarrows \mathbb{R}^2$,

$$F(x) = \begin{cases} \{0\} \times [0, 1], & \text{if } x = 0, \\ \left[(0, 0), \left(x, \frac{1}{x}\right)\right], & \text{if } x \in (0, 1], \\ \emptyset, & \text{if } x \in [-1, 0) \end{cases}$$

and

$$K(x) = \begin{cases} \mathbb{R}^2_+, & \text{if } x = 0, \\ \{(0, z) \mid \ z \leq 0\}, & \text{if } x \in [-1, 1] \setminus \{0\}, \end{cases}$$

where $[(a, b), (c, d)]$ is the line segment joining (a, b) and (c, d). It is easy to see that $(0, (0, 0))$ is local nondominated point for F with respect to K, but is not a local robust efficient point for F with respect to K.

It is known that a point is local (weakly) nondominated for F with respect to K if and only if that point is local (weakly) nondominated for $F + K$ with respect to K. In the following lemma we can see that we do not have this equivalence relation for the notion defined above, but we can get a weaker one. In order to illustrate this, we consider Example 2.8.

Lemma 2.7 *Take $(\overline{x},\overline{y}) \in \operatorname{Gr} F$. Then $(\overline{x},\overline{y})$ is a local robust efficient point for F with respect to K iff there exists a neighborhood U of $\overline{x}$, such that for every x, $z \in U$,*

$$(F(x) + K(z) - \overline{y}) \cap (-K(z)) \subset \{0\}.$$

Furthermore, if we suppose that there exists a neighborhood U of $\overline{x}$ such that $\operatorname{int} K(x) \neq \emptyset$ for all $x \in U$, then $(\overline{x},\overline{y})$ is a local robust weakly efficient point for F with respect to K iff there exists a neighborhood $V \subset U$ of $\overline{x}$, such that for every x, $z \in V$,

$$(F(x) + K(z) - \overline{y}) \cap (-\operatorname{int} K(z)) = \emptyset.$$

Proof. To prove this, we take $a \in (F(x) + K(z) - \overline{y}) \cap (-K(z))$, so there exists $k \in K(z)$ such that

$$a - k \in (F(x) - \overline{y}) \cap (-K(z) - K(z)) \subset (F(x) - \overline{y}) \cap (-K(z)) \subset \{0\},$$

which gives us $a = k \in K(z)$. It follows that $a \in K(z) \cap (-K(z))$ and as $K(x)$ is pointed for any $x \in X$, we obtain that $a = 0$. The other implication simply follows from the inclusion $(F(x) - \overline{y}) \cap (-K(z)) \subset (F(x) + K(z) - \overline{y}) \cap (-K(z))$. If one follows similar arguments as above one gets the second equivalence. $\square$

Example 2.8 If $(\overline{x},\overline{y})$ is a local robust (weakly) efficient point for F with respect to K, then $(\overline{x},\overline{y})$ is not necessarily a local robust (weakly) efficient point for $F + K$ with respect to K. For example, we take X, Y and F as in Example 2.6 and $K : X \rightrightarrows Y$,

$$K(x) = \begin{cases} \mathbb{R}^2_+, \text{ if } x = 0, \\ \operatorname{cone}\operatorname{conv}\{(-1,0),(1,1)\}, \text{ if } x \in [-1,1] \setminus \{0\}, \end{cases}$$

where cone and conv denote the conic and the convex hulls, respectively. It is easy to see that $(0,(0,0))$ is a local robust efficient point for F with respect to K. However, $(0,(0,0))$ it is not a local robust efficient point for $F + K$ with respect to K because for every neighborhood U of 0, there exist $0 \neq x \in U$, $z = 0$ such that $(-1,0) \in -K(z) \cap (F(x) + K(x))$.

3 Tools of generalized differentiation

In order to get optimality conditions for the notions introduced in Definitions 2.1 and 2.4 we use the constructions of generalized differentiation developed by Mordukhovich and his collaborators. Thus, following the book [15], we give some of the constructions that we use in this paper.

Definition 3.1 *(i) Let X be a normed vector space, S a nonempty subset of X, $x \in S$ and $\varepsilon \geq 0$. The set of $\varepsilon-$normals to S at x is*

$$\widehat{N}_\varepsilon(S,x) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{S} x} \frac{x^*(u-x)}{\|u-x\|} \leq \varepsilon \right\}, \tag{3.1}$$

where $u \xrightarrow{S} x$ means that $u \to x$ and $u \in S$.

If $\varepsilon = 0$, the elements in the right-hand side of (3.1) are called Fréchet normals and their collection, denoted by $\widehat{N}(S,x)$, is the Fréchet normal cone to S at x.

Let $\overline{x} \in S$. The basic (or limiting, or Mordukhovich) normal cone to S at $\overline{x}$ is

$$N(S, \overline{x}) := \{x^* \in X^* \mid \exists \varepsilon_n \downarrow 0, x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}_{\varepsilon_n}(S, x_n), \forall n \in \mathbb{N}\}.$$

(ii) Let $f : X \to \overline{\mathbb{R}}$, finite at $\overline{x} \in X$. The Fréchet subdifferential of f at $\overline{x}$ is the set

$$\widehat{\partial} f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\}$$

and the basic (or limiting, or Mordukhovich) subdifferential of f at $\overline{x}$ is

$$\partial f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in N(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\}.$$

If f is convex, then both $\partial f(\overline{x})$, $\widehat{\partial} f(\overline{x})$ do coincide with the classical Fenchel subdifferential. If X is an Asplund space (i.e., a Banach space where every convex continuous function is generically Fréchet differentiable), and S is closed around $\overline{x}$ (i.e., there is a neighborhood V of $\overline{x}$ such that $S \cap V$ is closed), the formula for the basic normal cone looks as follows:

$$N(S, \overline{x}) = \{x^* \in X^* \mid \exists x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}(S, x_n), \forall n \in \mathbb{N}\}.$$

Also, in general, one has the inclusion $\widehat{\partial} f(\overline{x}) \subset \partial f(\overline{x})$. A well known result is the next generalized Fermat rule: if $\overline{x}$ is a local minimum point for f then $0 \in \widehat{\partial} f(\overline{x})$. Let $A \subset X$ and $B \subset Y$ be nonempty sets. For an arbitrary point $(a, b) \in A \times B$ we have that $\widehat{N}(A \times B, (a, b)) = \widehat{N}(A, a) \times \widehat{N}(B, b)$ and $N(A \times B, (a, b)) = N(A, a) \times N(B, b)$, and for any $\overline{x} \in A$ one has

$$\widehat{\partial} d(\cdot, A)(\overline{x}) = \widehat{N}(A, \overline{x}) \cap D_{X^*}. \tag{3.2}$$

If δ_A denotes the indicator function associated with the set A, i.e.,

$$\delta_A(x) = \begin{cases} 0, & \text{if } x \in A, \\ \infty, & \text{otherwise,} \end{cases}$$

then $\widehat{\partial} \delta_A(\overline{x}) = \widehat{N}(A, \overline{x})$ and $\partial \delta_A(\overline{x}) = N(A, \overline{x})$ for any $\overline{x} \in A$.

Further, we recall the fuzzy sum rule and a composition rule for the Fréchet subdifferential (see [15, Theorem 2.33] and [16, Theorem 3.7], respectively).

Theorem 3.2 *Let X be an Asplund space and $\varphi_1, \varphi_2 : X \to \mathbb{R} \cup \{+\infty\}$ be such that φ_1 is Lipschitz continuous around $\overline{x} \in \operatorname{dom} \varphi_1 \cap \operatorname{dom} \varphi_2$ and φ_2 is lower semicontinuous around $\overline{x}$. Then for any $\gamma > 0$ one has*

$$\widehat{\partial}(\varphi_1 + \varphi_2)(\overline{x}) \subset \bigcup \left\{ \widehat{\partial} \varphi_1(x_1) + \widehat{\partial} \varphi_2(x_2) \mid \ x_i \in \overline{x} + \gamma D_X, \ \ |\varphi_i(x_i) - \varphi_i(\overline{x})| \leq \gamma, \ i = 1, 2 \right\} + \gamma D_{X^*}.$$

Proposition 3.3 *Let $f : X \to Y$ be a Lipschitz function and let $\varphi : Y \to \mathbb{R}$. Let $\overline{x} \in X$ and suppose that φ is Fréchet differentiable at $f(\overline{x})$. Then*

$$\widehat{\partial}(\varphi \circ f)(\overline{x}) = \widehat{\partial}(\nabla \varphi(f(\overline{x})) \circ f)(\overline{x}).$$

In the following, we present two concepts of coderivatives for set-valued maps.

Definition 3.4 *Let $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Then the Fréchet coderivative of F at $(\overline{x}, \overline{y})$ is the set-valued map $\widehat{D}^* F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ given by*

$$\widehat{D}^* F(\overline{x}, \overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}(\operatorname{Gr} F, (\overline{x}, \overline{y}))\}.$$

Similarly, the normal coderivative of F at $(\overline{x}, \overline{y})$ is the set-valued map $D_N^* F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ given by

$$D_N^* F(\overline{x}, \overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N(\operatorname{Gr} F, (\overline{x}, \overline{y}))\}.$$

Next we recall a technical result (see [6, Lemma 4.9]).

Proposition 3.5 *Let $K : X \rightrightarrows Y$ be a set-valued map with $K(x)$ a convex cone for any $x \in X$ and $(\overline{x}, \overline{y}) \in \operatorname{Gr} K$.*

(i) If $\widehat{D}^* K(\overline{x}, \overline{y})(y^*) \neq \emptyset$, then $y^* \in (K(\overline{x}))^*$.

(ii) If X and Y are Asplund spaces, $\operatorname{Gr} K$ is closed around $(\overline{x}, \overline{y})$, K is lower semicontinuous at $\overline{x}$ (i.e., for every $y \in K(\overline{x})$ and every sequence $(x_n) \to \overline{x}$, there exists a sequence $(y_n) \to y$ with $(x_n, y_n) \in \operatorname{Gr} K$ for every n) and $D_N^* K(\overline{x}, \overline{y})(y^*) \neq \emptyset$, then $y^* \in (K(\overline{x}))^*$.

Further, we recall the Lipschitz-like property for set-valued maps and a version of coderivative sum rules for mappings (see [15, Proposition 3.12]).

Definition 3.6 *Let $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. The set-valued mapping F is Lipschitz-like around $(\overline{x}, \overline{y})$ if there exist a constant $M > 0$ and some neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ such that, for every $x, u \in U$,*

$$F(x) \cap V \subset F(u) + M \|x - u\| D_Y.$$

Proposition 3.7 *Let X, Y be Asplund spaces, $S \subset X$ be a closed set and the set-valued map $F : X \rightrightarrows Y$ be closed around $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. If F is Lipschitz-like around $(\overline{x}, \overline{y})$, then for every $y^* \in Y^*$, we have the inclusion*

$$D_N^* F_S(\overline{x}, \overline{y})(y^*) \subset D_N^* F(\overline{x}, \overline{y})(y^*) + N(S, \overline{x}),$$

where $F_S(x) := F(x)$, if $x \in S$ and $F_S(x) := \emptyset$, if $x \notin S$.

Moreover, we need as well a calculus rule for the Fréchet normal cone of the intersection of a finite number of sets. Given the closed subsets $C_1, ..., C_k$ of a normed vector space X, one says that they are allied at $\overline{x} \in C_1 \cap ... \cap C_k$ whenever $(x_{in}) \xrightarrow{C_i} \overline{x}$, $x_{in}^* \in \widehat{N}(C_i, x_{in})$, $i = \overline{1,k}$, the relation $(x_{1n}^* + ... + x_{kn}^*) \to 0$ implies $(x_{in}^*) \to 0$ for every $i = \overline{1,k}$. The concept of alliedness was introduced by Penot in [19]. By Theorem 4.1 from [4], the alliedness of the subsets $C_1, ..., C_k$ implies that there exists $r > 0$ such that, for every $\varepsilon > 0$ and every $x \in [C_1 \cap ... \cap C_k] \cap B(\overline{x}, r)$, there exist $x_i \in C_i \cap B(x, \varepsilon)$, $i = \overline{1,k}$ such that

$$\widehat{N}(C_1 \cap ... \cap C_k, x) \subset \widehat{N}(C_1, x_1) + ... + \widehat{N}(C_k, x_k) + \varepsilon D_{X^*}.$$

Now, we are able to separately study, in the next sections, the notions defined in Section 2. However, we will see that the treatment of both cases has several common points (see the last section).

4 Optimality conditions for sharp nondominated points

Let F, $K : X \rightrightarrows Y$ be the set-valued maps considered in Section 2, that is the objective mapping and the ordering mapping, respectively. We work with the following sets

$$C_1 := \{(x, y, z) \in X \times Y \times Y \mid y \in F(x)\},$$
$$C_2 := \{(x, y, z) \in X \times Y \times Y \mid z \in K(x)\}.$$

With this notation, we formulate our first result, in which we obtained necessary optimality conditions in terms of Fréchet coderivatives for the notion of sharp efficiency, as follows.

Theorem 4.1 *Let X, Y be Asplund spaces, F, $K : X \rightrightarrows Y$ be two set-valued maps with $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ and $(\overline{x}, 0) \in \operatorname{Gr} K$ such that $\operatorname{Gr} F$, $\operatorname{Gr} K$ are closed around $(\overline{x}, \overline{y})$ and $(\overline{x}, 0)$, respectively. Assume that the following assumptions are satisfied:*

(i) $(\overline{x}, \overline{y})$ is a weak ψ-sharp local nondominated point (with the constant $\alpha > 0$) for the problem (P);

(ii) the sets C_1 and C_2 are allied at $(\overline{x}, \overline{y}, 0)$;

(iii) ψ is Fréchet differentiable at 0 with $\nabla\psi(0) > 0$.

Then for every $t^* \in \alpha\nabla\psi(0) D_{X^*} \cap \widehat{N}(W, \overline{x})$, and for every $\varepsilon > 0$, there exist $(x_1, y_1) \in \operatorname{Gr} F \cap (B(\overline{x}, \varepsilon) \times B(\overline{y}, \varepsilon))$, $(x_2, y_2) \in \operatorname{Gr} K \cap (B(\overline{x}, \varepsilon) \times B(0, \varepsilon))$, $y^* \in D_{Y^*}$ such that

$$t^* \in \widehat{D}^* F(x_1, y_1)(y^* + \varepsilon B_{Y^*}) + \widehat{D}^* K(x_2, y_2)(y^* + \varepsilon B_{Y^*}) + \varepsilon B_{X^*}.$$

Proof. Using Definition 2.1, there exist $\alpha > 0$ and a neighborhood U of $\overline{x}$ such that for every $x \in U$, $y \in F(x)$, $z \in K(x)$ one has

$$\alpha\psi(d(x, W)) \leq d(y - \overline{y}, -K(x)) \leq \|y - \overline{y} + z\|.$$

Whence, $(\overline{x}, \overline{y}, 0)$ is a minimum point for the function

$$X \times Y \times Y \ni (x, y, z) \mapsto \|y - \overline{y} + z\| - \alpha\psi(d(x, W)) \in \mathbb{R}$$

on $(U \times Y \times Y) \cap (C_1 \cap C_2)$. Using the infinite penalization, we obtain that $(\overline{x}, \overline{y}, 0)$ is a local minimum point (without constraints) for the function $g : X \times Y \times Y \to \overline{\mathbb{R}}$ given by

$$g(x, y, z) := \|y - \overline{y} + z\| - \alpha\psi(d(x, W)) + \delta_{C_1 \cap C_2}(x, y, z).$$

It results, from the generalized Fermat rule, that

$$(0, 0, 0) \in \widehat{\partial} g(\overline{x}, \overline{y}, 0).$$

Let $g_1 : X \times Y \times Y \to \mathbb{R}$ and $g_2 : X \times Y \times Y \to \overline{\mathbb{R}}$ be the functions given by $g_1(x, y, z) := \alpha\psi(d(x, W))$ and $g_2(x, y, z) := \|y - \overline{y} + z\| + \delta_{C_1 \cap C_2}(x, y, z)$, respectively. Using relation (3.2) and Proposition 3.3, one observes that

$$\widehat{\partial} g_1(\overline{x}, \overline{y}, 0) = \left[\alpha\nabla\psi(0) D_{X^*} \cap \widehat{N}(W, \overline{x})\right] \times \{0\} \times \{0\} \supset \{(0, 0, 0)\},$$

so $\widehat{\partial} g_1(\overline{x}, \overline{y}, 0) \neq \emptyset$. It follows from [16, Theorem 3.1] that

$$(0, 0, 0) \in \bigcap_{t^* \in \alpha\nabla\psi(0) D_{X^*} \cap \widehat{N}(W, \overline{x})} \left[\widehat{\partial} g_2(\overline{x}, \overline{y}, 0) - (t^*, 0, 0)\right]. \tag{4.1}$$

Taking into account that g_2 is the sum between a Lipschitz function and a lower semicontinuous one around $(\overline{x}, \overline{y}, 0)$, we can apply the fuzzy calculus rule for the Fréchet subdifferential. Thus, for every $\varepsilon > 0$, there exist

$$(x_1, y_1, z_1) \in B\left(\overline{x}, \frac{\varepsilon}{2}\right) \times B\left(\overline{y}, \frac{\varepsilon}{2}\right) \times B\left(0, \frac{\varepsilon}{2}\right)$$

and

$$(x_2, y_2, z_2) \in \left[B\left(\overline{x}, \frac{\varepsilon}{2}\right) \times B\left(\overline{y}, \frac{\varepsilon}{2}\right) \times B\left(0, \frac{\varepsilon}{2}\right)\right] \cap (C_1 \cap C_2)$$

such that

$$\widehat{\partial} g_2(\overline{x}, \overline{y}, 0) \subset \{0\} \times \partial \|\cdot + \cdot - \overline{y}\| (y_1, z_1) + \widehat{\partial} \delta_{C_1 \cap C_2}(x_2, y_2, z_2) + \frac{\varepsilon}{2}(B_{X^*} \times B_{Y^*} \times B_{Y^*}). \quad (4.2)$$

Notice that the closedness assumption of graphs of F and K implies that C_1 and C_2 are closed around $(\overline{x}, \overline{y}, 0)$. Using the hypothesis (ii) we obtain that

$$\begin{aligned} \widehat{\partial} \delta_{C_1 \cap C_2}(x_2, y_2, z_2) &= \widehat{N}(C_1 \cap C_2, (x_2, y_2, z_2)) \\ &\subset \widehat{N}(C_1, (x_{21}, y_{21}, z_{21})) + \widehat{N}(C_2, (x_{22}, y_{22}, z_{22})) + \frac{\varepsilon}{2}(D_{X^*} \times D_{Y^*} \times D_{Y^*}), \quad (4.3) \end{aligned}$$

where

$$(x_{21}, y_{21}, z_{21}) \in \left[B\left(x_2, \frac{\varepsilon}{2}\right) \times B\left(y_2, \frac{\varepsilon}{2}\right) \times B\left(z_2, \frac{\varepsilon}{2}\right)\right] \cap C_1$$

and

$$(x_{22}, y_{22}, z_{22}) \in \left[B\left(x_2, \frac{\varepsilon}{2}\right) \times B\left(y_2, \frac{\varepsilon}{2}\right) \times B\left(z_2, \frac{\varepsilon}{2}\right)\right] \cap C_2.$$

Now, defining the linear operator $A : Y \times Y \to Y$ by $A(y, z) := y + z$, the function $(y, z) \mapsto \|y + z - \overline{y}\|$ can be expressed as the composition between the convex function $y \mapsto \|y - \overline{y}\|$ and A. Applying [20, Theorem 2.8.6], we have that

$$\partial \|\cdot + \cdot - \overline{y}\| (y_1, z_1) = A^* (\partial \|\cdot - \overline{y}\|)(y_1 + z_1),$$

where $A^* : Y^* \to Y^* \times Y^*$ denotes the adjoint of A. As $A^*(y^*) = (y^*, y^*)$ for every $y^* \in Y^*$, we obtain that

$$\partial \|\cdot + \cdot - \overline{y}\| (y_1, z_1) \subset \{(y^*, y^*) \mid y^* \in D_{Y^*}\}.$$

Fix $t^* \in \alpha \nabla \psi(0) D_{X^*} \cap \widehat{N}(W, \overline{x})$. Thus, from (4.1), (4.2) and (4.3) we obtain that there exist $y^* \in D_{Y^*}$,

$$\begin{aligned} (x_1^*, -y_1^*, 0) \in \widehat{N}(C_1, (x_{21}, y_{21}, z_{21})) &\Leftrightarrow x_1^* \in \widehat{D}^* F(x_{21}, y_{21})(y_1^*), \\ (x_2^*, 0, -z_2^*) \in \widehat{N}(C_2, (x_{22}, y_{22}, z_{22})) &\Leftrightarrow x_2^* \in \widehat{D}^* K(x_{22}, z_{22})(z_2^*), \end{aligned}$$

such that

$$\begin{aligned} t^* &\in x_1^* + x_2^* + \varepsilon D_{X^*}, \\ y_1^* &\in y^* + \varepsilon D_{Y^*}, \\ z_2^* &\in y^* + \varepsilon D_{Y^*}. \end{aligned}$$

Since $(x_{21}, y_{21}) \in \operatorname{Gr} F \cap [B(\overline{x}, \varepsilon) \times B(\overline{y}, \varepsilon)]$ and $(x_{22}, z_{22}) \in \operatorname{Gr} K \cap [B(\overline{x}, \varepsilon) \times B(0, \varepsilon)]$, we have the conclusion. $\square$

Remark 4.2 *Taking into account the special form of C_1 and C_2, the alliedness at $(\overline{x},\overline{y},0)$ means that for any sequences $(x_n,y_n) \overset{\mathrm{Gr}\,F}{\to} (\overline{x},\overline{y})$, $(u_n,v_n) \overset{\mathrm{Gr}\,K}{\to} (\overline{x},0)$, and every $x_n^* \in \widehat{D}^*F(x_n,y_n)(y_n^*)$, $u_n^* \in \widehat{D}^*K(u_n,v_n)(v_n^*)$, $(x_n^*+u_n^*) \to 0$, $y_n^* \to 0$, $v_n^* \to 0$ imply that $x_n^* \to 0$ and $u_n^* \to 0$. According to [15, Theorem 1.43], the sets C_1 and C_2 are allied at $(\overline{x},\overline{y},0)$ if F is Lipschitz-like around $(\overline{x},\overline{y}) \in \mathrm{Gr}\,F$ or K is Lipschitz-like around $(\overline{x},0) \in \mathrm{Gr}\,K$.*

Remark 4.3 *Notice that it is possible to obtain optimality conditions by considering, as usual, the epigraphical set-valued map associated to F, but in this case of variable ordering structure we consider that this approach is not appropriate since it would lead to optimality conditions in terms of coderivative of the sum $F+K$, which in this situation is more complicated than the sum of the coderivatives of F and K, respectively. So, in this sense the result above and the following ones are, in our opinion, adapted to this specific situation for which the alliedness condition is essential.*

To obtain optimality conditions in terms of Mordukhovich coderivatives we use the fact that the closed unit ball of X^* is weak* sequentially compact, if X is an Asplund space (see [15, Proposition 1.123]).

Theorem 4.4 *Let X, Y be Asplund spaces, F, $K : X \rightrightarrows Y$ be two set-valued maps with $(\overline{x},\overline{y}) \in \mathrm{Gr}\,F$ and $(\overline{x},0) \in \mathrm{Gr}\,K$ such that $\mathrm{Gr}\,F$, $\mathrm{Gr}\,K$ are closed around $(\overline{x},\overline{y})$ and $(\overline{x},0)$, respectively. Assume that the following assumptions are satisfied:*

(i) $(\overline{x},\overline{y})$ is a weak ψ-sharp local nondominated point (with the constant $\alpha > 0$) for the problem (P);

(ii) the sets C_1 and C_2 are allied at $(\overline{x},\overline{y},0)$;

(iii) ψ is Fréchet differentiable at 0 with $\nabla\psi(0) > 0$;

(iv) K is lower semicontinuous at $\overline{x}$.

Then for every $t^* \in \alpha\nabla\psi(0)\,D_{X^*} \cap \widehat{N}(W,\overline{x})$, there exists $y^* \in D_{Y^*} \cap (K(\overline{x}))^*$ such that

$$t^* \in D_N^*F(\overline{x},\overline{y})(y^*) + D_N^*K(\overline{x},0)(y^*). \tag{4.4}$$

Proof. Let $t^* \in \alpha\nabla\psi(0)\,D_{X^*} \cap \widehat{N}(W,\overline{x})$. It follows from the previous theorem, that there exist $(x_n,y_n) \overset{\mathrm{Gr}\,F}{\to} (\overline{x},\overline{y})$, $(u_n,v_n) \overset{\mathrm{Gr}\,K}{\to} (\overline{x},0)$, $(y_n^*) \subset D_{Y^*}$, (t_n^*), $(v_n^*) \to 0$ and (x_n^*), $(u_n^*) \subset X^*$ such that $x_n^* \in \widehat{D}^*F(x_n,y_n)(y_n^*+t_n^*)$, $u_n^* \in \widehat{D}^*K(u_n,v_n)(y_n^*+v_n^*)$ for every positive integer n and $x_n^*+u_n^* \to t^*$. Since, (y_n^*) is bounded, we can suppose, without loosing the generality, that it converges weakly* to an element y^*. Therefore $y_n^*+v_n^*$, $y_n^*+t_n^* \overset{w^*}{\to} y^*$, so $(y_n^*+v_n^*)$, $(y_n^*+t_n^*)$ are bounded. Further, we want to prove that the sequences (x_n^*) and (u_n^*) are bounded. As $x_n^*+u_n^* \to t^*$, it is sufficient to prove that one of these two sequences is bounded. Suppose by contradiction that both sequences are unbounded. It follows that for every n, there is k_n sufficiently large such that

$$n < \min\left\{\left\|x_{k_n}^*\right\|, \left\|u_{k_n}^*\right\|\right\}. \tag{4.5}$$

For simplicity we denote the subsequences $\left(x_{k_n}^*\right)$, $\left(u_{k_n}^*\right)$ by (x_n^*), (u_n^*), respectively. Now, using the positive homogeneity of the Fréchet coderivatives, we obtain that

$$\begin{aligned}
\frac{1}{n}x_n^* &\in \widehat{D}^*F(x_n,y_n)\left(\frac{1}{n}(y_n^*+t_n^*)\right), \\
\frac{1}{n}u_n^* &\in \widehat{D}^*K(u_n,v_n)\left(\frac{1}{n}(y_n^*+v_n^*)\right).
\end{aligned}$$

As $\frac{1}{n}(y_n^*+t_n^*) \to 0$, $\frac{1}{n}(y_n^*+v_n^*) \to 0$ and $\frac{1}{n}x_n^*+\frac{1}{n}u_n^* \to 0$, we have from the alliedness of the sets C_1, C_2 that $\frac{1}{n}x_n^* \to 0$ and $\frac{1}{n}u_n^* \to 0$, which is in contradiction with relation (4.5). Consequently, we obtain that (x_n^*), (u_n^*) are bounded, whence there exist x^*, $u^* \in X^*$ such that $x_n^* \xrightarrow{w^*} x^*$ and $u_n^* \xrightarrow{w^*} u^*$ (without relabelling). Using again that $x_n^*+u_n^* \to t^*$, we deduce that $t^*=x^*+u^*$. It follows that $x^* \in D_N^*F(\overline{x},\overline{y})(y^*)$, $u^* \in D_N^*K(\overline{x},0)(y^*)$, with $y^* \in D_{Y^*}$. Now, using Proposition 3.5 (ii) we obtain that $y^* \in (K(\overline{x}))^*$, which gives the conclusion of the theorem. □

Remark 4.5 *As we can see, in the above theorem we do not know if the multiplier y^* is nonzero, so we get a result of Fritz John type.*

In the next result we suppose that the set-valued map K is constantly equal to a closed convex proper cone Q. Notice that if we suppose that the epigraphical set-valued map associated to F is closed around $(\overline{x},\overline{y}) \in \operatorname{Gr} F$, using Remark 2.2 (ii), we obtain Theorem 4.1 from [5].

Corollary 4.6 *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a set-valued map with $\operatorname{Gr} F$ closed around $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ and Q be a closed convex proper cone. Assume that $(\overline{x},\overline{y})$ is a weak ψ-sharp local nondominated point (with the constant $\alpha > 0$) for the problem (P), ψ is Fréchet differentiable at 0 with $\nabla\psi(0) > 0$. Then for every $t^* \in \alpha\nabla\psi(0) D_{X^*} \cap \widehat{N}(W,\overline{x})$, there exists $y^* \in D_{Y^*} \cap Q^*$ such that*

$$t^* \in D_N^*F(\overline{x},\overline{y})(y^*).$$

Proof. Consider the set-valued map $K : X \rightrightarrows Y$ such that $K(x) := Q$ for all $x \in X$. The sets C_1, C_2 are allied at $(\overline{x},\overline{y},0)$ if for any sequences $(x_n,y_n) \xrightarrow{\operatorname{Gr} F} (\overline{x},\overline{y})$, $(u_n,v_n) \xrightarrow{\operatorname{Gr} K} (\overline{x},0)$ and every $x_n^* \in \widehat{D}^*F(x_n,y_n)(y_n^*)$, $u_n^* \in \widehat{D}^*K(u_n,v_n)(v_n^*)$ such that $(y_n^*) \to 0$, $(v_n^*) \to 0$, $(x_n^*+u_n^*) \to 0$ we get $x_n^* \to 0$ and $u_n^* \to 0$. As $\widehat{D}^*K(u_n,v_n)(v_n^*) = \{0\}$, the alliedness of C_1 and C_2 at $(\overline{x},\overline{y},0)$ is proved. Also, it is easy to see that K is lower semicontinuous at $\overline{x}$ and $D_N^*K(\overline{x},0)(y^*) = \{0\}$. So, as all the hypotheses from the previous theorem are satisfied, we get the conclusion. □

Of course, in Theorem 4.4, if $0 \in D_N^*K(\overline{x},0)(y^*)$, then it becomes a consequence of Corollary 4.6.

Now, in order to get a Karush-Kuhn-Tucker type result, we restrict ourselves to the following particular case: we consider, instead of the set-valued map F, a single-valued map, denoted f, and instead of weak sharp efficiency a genuine sharp efficiency, that is $W = \{\overline{x}\}$. We denote now the problem (P_S) by $(\widetilde{P}_S)$ in order to mark the announced differences.

In order to get optimality conditions for the sharp efficiency notion, we use the Gerstewitz's (Tammer's) scalarizing functional. Thus, in the next lemma we collect from [1, Theorem 2.3.1, Corollary 2.3.5] and [7, Lemma 2.4] the properties of this nonlinear scalarization function which we need in the sequel.

Lemma 4.7 *Let $K \subset Y$ be a closed convex cone with nonempty interior. Then for every $e \in \operatorname{int} K$ the functional $s_{e,K} : Y \to \mathbb{R}$ given by*

$$s_{e,K}(y) := \inf\{\lambda \in \mathbb{R} \mid \lambda e \in y + K\} \tag{4.6}$$

is convex, continuous and for every $\lambda \in \mathbb{R}$

$$\{y \in Y \mid s_{e,K}(y) < \lambda\} = \lambda e - \operatorname{int} K, \tag{4.7}$$

$$\{y \in Y \mid s_{e,K}(y) \le \lambda\} = \lambda e - K. \tag{4.8}$$

Moreover, $s_{e,K}$ is sublinear, for every $u \in Y$, $\partial s_{e,K}(u)$ is nonempty and

$$\partial s_{e,K}(u) = \{v^* \in K^* \mid v^*(e) = 1, v^*(u) = s_{e,K}(u)\},$$

where ∂ denotes the Fenchel subdifferential. In addition, $s_{e,K}$ is $(d(e, \operatorname{bd} K))^{-1}$-Lipschitzian and for every $u \in Y$ and $v^* \in \partial s_{e,K}(u)$ one has

$$\|e\|^{-1} \leq \|v^*\| \leq (d(e, \operatorname{bd} K))^{-1}.$$

Next, we prove a result which show that the above function is appropriate to investigate sharp nondominated points.

Lemma 4.8 *Let $(\overline{x}, f(\overline{x}))$ be a ψ-sharp local nondominated point (with the neighborhood U and with the constant $\alpha > 0$) for the problem $\left(\widetilde{P}_S\right)$ and let $\overline{K} := \bigcap_{x \in U} K(x)$. If $\operatorname{int} \overline{K} \neq \emptyset$, then for every $e \in \operatorname{int} \overline{K}$, there exists $\mu > 0$ such that for every $x \in U \cap S$ and $z \in K(x)$ one has*

$$s_{e,\overline{K}}(z + f(x) - f(\overline{x})) \geq \mu \psi(\|x - \overline{x}\|). \tag{4.9}$$

Proof. By Definition 2.1 applied for $\left(\widetilde{P}_S\right)$ (that is, $F := f$ and $W = \{\overline{x}\}$), we get that there exist $\alpha > 0$ and a neighborhood U of $\overline{x}$ such that for every $x \in U \cap S$ one has

$$d(f(x) - f(\overline{x}), -K(x)) \geq \alpha \psi(\|x - \overline{x}\|). \tag{4.10}$$

Take $e \in \operatorname{int} \overline{K}$ with $\|e\| = 1$ and fix $x \in U \cap S$ and $z \in K(x)$. There are two possible cases.

First, if $x = \overline{x}$ then $\psi(\|x - \overline{x}\|) = 0$ and $s_{e,\overline{K}}(z + f(x) - f(\overline{x})) = s_{e,\overline{K}}(z)$. Now, if $z = 0$ then $z \notin -\operatorname{int} \overline{K}$. Next, we assume $z \in K(x) \setminus \{0\}$. As $K(x)$ is pointed and $z \in K(x)$, we obtain that $z \notin -K(x)$, so $z \notin -\operatorname{int} \overline{K}$. Using relation (4.7), we get that $s_{e,\overline{K}}(z) \geq 0$, i.e., the conclusion for any $\mu > 0$.

In the second case we suppose that $x \neq \overline{x}$. We want to prove that for any $\gamma \in (0, \alpha)$ one has

$$z + f(x) - f(\overline{x}) \notin \gamma \psi(\|x - \overline{x}\|) e - \operatorname{int} \overline{K}.$$

Suppose that there exists $\gamma \in (0, \alpha)$ such that $z + f(x) - f(\overline{x}) \in \gamma \psi(\|x - \overline{x}\|) e - \operatorname{int} \overline{K}$. Since $z \in K(x)$, we obtain that $f(x) - f(\overline{x}) \in \gamma \psi(\|x - \overline{x}\|) e - \operatorname{int} K(x)$. Then, using the fact that $\psi(\|x - \overline{x}\|) > 0$, we have

$$d(f(x) - f(\overline{x}), -K(x)) \leq \|\gamma \psi(\|x - \overline{x}\|) e\| < \alpha \psi(\|x - \overline{x}\|),$$

which is a contradiction with the inequality (4.10). Using again relation (4.7), we obtain that $s_{e,\overline{K}}(z + f(x) - f(\overline{x})) \geq \gamma \psi(\|x - \overline{x}\|)$.

Finally, take $e_1 \in \operatorname{int} \overline{K}$ arbitrarily. It's easy to see that $s_{e_1,\overline{K}}(\cdot) = \frac{1}{\|e_1\|} s_{e,\overline{K}}(\cdot)$, where $e = \frac{e_1}{\|e_1\|}$. It follows that we have the conclusion in every of the previous cases for x and z fixed before. Consequently, as the constant obtained does not depend of x or z, we obtain the conclusion. $\square$

In the following we consider as well, besides the set C_2, the set

$$C_3 := \{(x, y, z) \in X \times Y \times Y \mid y = f(x)\}.$$

Theorem 4.9 *Let X, Y be Asplund spaces, $f : X \to Y$ be a function continuous around $\overline{x}$ with $\overline{y} = f(\overline{x})$ and $K : X \rightrightarrows Y$ be a set-valued map such that $\operatorname{Gr} K$ is closed around $(\overline{x}, 0) \in \operatorname{Gr} K$. Assume that the following assumptions are satisfied:*

(i) $(\overline{x}, \overline{y})$ is a ψ-sharp local nondominated point (with the neighborhood U) for the problem $\left(\widetilde{P}\right)$;

(ii) the sets C_2 and C_3 are allied at $(\overline{x}, \overline{y}, 0)$;

(iii) ψ is Fréchet differentiable at 0 with $\nabla\psi(0) > 0$ and $\operatorname{int} \overline{K} \neq \emptyset$, where $\overline{K} = \bigcap_{x \in U} K(x)$.

Then for $e \in \operatorname{int} \overline{K}$, there exists $\mu > 0$ such that for every $t^* \in \mu\nabla\psi(0) D_{X^*}$, and every $\varepsilon > 0$, there exist $x_1 \in B(\overline{x}, \varepsilon)$, $(x_2, y_2) \in \operatorname{Gr} K \cap (B(\overline{x}, \varepsilon) \times B(0, \varepsilon))$, $y \in B(0, \varepsilon)$, $y^* \in \partial s_{e,\overline{K}}(y)$ with

$$t^* \in \widehat{D}^* f(x_1)(y^* + \varepsilon B_{Y^*}) + \widehat{D}^* K(x_2, y_2)(y^* + \varepsilon B_{Y^*}) + \varepsilon B_{X^*}.$$

Proof. Using Lemma 4.8, we obtain that for every $e \in \operatorname{int} \overline{K}$, there exist $\mu > 0$ and a neighborhood U of $\overline{x}$ such that for every $x \in U$, $y = f(x)$, $z \in K(x)$

$$s_{e,\overline{K}}(y + z - \overline{y}) \geq \mu\psi(\|x - \overline{x}\|).$$

Whence, $(\overline{x}, \overline{y}, 0)$ is a minimum point for the function

$$X \times Y \times Y \ni (x, y, z) \mapsto s_{e,\overline{K}}(y + z - \overline{y}) - \mu\psi(\|x - \overline{x}\|) \in \mathbb{R}$$

on $(U \times Y \times Y) \cap (C_2 \cap C_3)$. In the following, we can use the same technique as in the proof of Theorem 4.1, with $s_{e,\overline{K}}(y + z - \overline{y})$ instead of $\|y - \overline{y} + z\|$, with the obvious modifications. The conclusion follows. $\square$

Theorem 4.10 *Let X, Y be Asplund spaces, $f : X \to Y$ be a function continuous around $\overline{x}$ with $\overline{y} = f(\overline{x})$ and $K : X \rightrightarrows Y$ be a set-valued map such that $\operatorname{Gr} K$ is closed around $(\overline{x}, 0) \in \operatorname{Gr} K$. Assume that the following assumptions are satisfied:*

(i) $(\overline{x}, \overline{y})$ is a ψ-sharp local nondominated point (with the neighborhood U) for the problem $\left(\widetilde{P}\right)$;

(ii) the sets C_2 and C_3 are allied at $(\overline{x}, \overline{y}, 0)$;

(iii) ψ is Fréchet differentiable at 0 with $\nabla\psi(0) > 0$ and $\operatorname{int} \overline{K} \neq \emptyset$, where $\overline{K} = \bigcap_{x \in U} K(x)$;

Then for $e \in \operatorname{int} \overline{K}$, there exists $\mu > 0$ such that for every $t^* \in \mu\nabla\psi(0) D_{X^*}$ there exists $y^* \in \overline{K}^*$ with $y^*(e) = 1$ and

$$t^* \in D_N^* f(\overline{x})(y^*) + D_N^* K(\overline{x}, 0)(y^*). \tag{4.11}$$

Proof. As all the assumptions from previous theorem are satisfied, it follows that for every $e \in \operatorname{int} \overline{K}$, there exists $\mu > 0$, for any $t^* \in \mu\nabla\psi(0) D_{X^*}$, there exist $(x_n) \to \overline{x}$, $(u_n, v_n) \overset{\operatorname{Gr} K}{\to} (\overline{x}, 0)$, $(y_n) \to 0$, $(y_n^*) \subset \partial s_{e,\overline{K}}(y_n)$, (t_n^*), $(v_n^*) \to 0$ and (x_n^*), $(u_n^*) \subset X^*$ such that $x_n^* \in \widehat{D}^* f(x_n)(y_n^* + t_n^*)$, $u_n^* \in \widehat{D}^* K(u_n, v_n)(y_n^* + v_n^*)$ for every positive integer n and $x_n^* + u_n^* \to t^*$. As $y_n^* \in \partial s_{e,\overline{K}}(y_n)$, we obtain from Lemma 4.7 that $\|e\|^{-1} \leq \|y_n^*\| \leq \left(d\left(e, \operatorname{bd} \overline{K}\right)\right)^{-1}$ and $(y_n^*) \subset \overline{K}^*$. Since, (y_n^*) is bounded, we can suppose, without loosing the generality, that it converges weakly* to an element y^*. Therefore $y_n^* + v_n^*$, $y_n^* + t_n^* \overset{w^*}{\to} y^*$, so $(y_n^* + v_n^*)$, $(y_n^* + t_n^*)$ are bounded. As in Theorem 4.4, we obtain that there exist x^*, u^* such that $t^* = x^* + u^*$. As $\overline{K}^*$ is weakly-star closed, $(y_n^*) \subset \overline{K}^*$ and $y_n^* \overset{w^*}{\to} y^*$, we get that $y^* \in \overline{K}^*$. Using again Lemma 4.7 and the fact that $\partial s_{e,\overline{K}}$ is w^*-closed, we have that $y^*(e) = 1$ and the conclusion follows. $\square$

Now, we briefly consider the case of problem $\left(\widetilde{P}_S\right)$. Notice that a similar result for the case where the objective is a set-valued map can be done in the same way.

Theorem 4.11 *Let X, Y be Asplund spaces, $f : X \to Y$ be a function continuous around $\overline{x}$ with $\overline{y} = f(\overline{x})$ and $K : X \rightrightarrows Y$ be a set-valued map such that* $\operatorname{Gr} K$ *is closed around* $(\overline{x}, 0) \in \operatorname{Gr} K$. *Suppose that $(\overline{x}, \overline{y})$ is a ψ-sharp local nondominated point (with the neighborhood U) for the problem $\left(\widetilde{P}_S\right)$, ψ is Fréchet differentiable at 0, $\nabla\psi(0) > 0$, S is closed and* $\operatorname{int} \overline{K} \neq \emptyset$, *where* $\overline{K} := \bigcap_{x \in U} K(x)$. *If f is Lipschitz around $\overline{x}$ or K is Lipschitz-like around $(\overline{x}, 0)$, then for $e \in \operatorname{int} \overline{K}$, there exists $\mu > 0$ such that for every $t^* \in \mu\nabla\psi(0) D_{X^*}$ there exists $y^* \in \overline{K}^*$ with $y^*(e) = 1$ and*

$$t^* \in D_N^* f(\overline{x})(y^*) + D_N^* K(\overline{x}, 0)(y^*) + N(S, \overline{x}).$$

Proof. Suppose that f is Lipschitz around $\overline{x}$ and take $e \in \operatorname{int} \overline{K}$. The conclusion follows analogously as in Theorems 4.9 and 4.10. Indeed, we use again Lemma 4.8 and the infinite penalization to obtain that there is $\mu > 0$ such that $(\overline{x}, \overline{y}, 0)$ is a local minimum point (without constraints) for the function $g : X \times Y \times Y \to \overline{\mathbb{R}}$ given by

$$g(x, y, z) := s_{e,\overline{K}}(y + z - \overline{y}) - \mu\psi(d(x, W)) + \delta_{C_2 \cap C_3^S}(x, y, z),$$

where $C_3^S := \{(x, y, z) \in S \times Y \times Y \mid y = f(x)\}$. According to Remark 4.2, the assumption (ii) from Theorem 4.10 adapted to this case is satisfied. As in the proof of the above mentioned theorems, we get that for every $t^* \in \mu\nabla\psi(0) D_{X^*}$, there exists $y^* \in \overline{K}^*$ such that $y^*(e) = 1$ and

$$t^* \in D_N^* f_S(\overline{x})(y^*) + D_N^* K(\overline{x}, 0)(y^*),$$

where $f_S(x) := f(x)$, if $x \in S$ and $f_S(x) := \emptyset$, if $x \notin S$. By Proposition 3.7, taking $F := f$, we get the conclusion. It is easy to see that $C_2 \cap C_3^S = C_2^S \cap C_3$, where $C_2^S := \{(x, y, z) \in S \times Y \times Y \mid z \in K(x)\}$. So, if we suppose that K is Lipschitz-like around $(\overline{x}, 0)$, the proof is the same. $\square$

Remark 4.12 *Analogously to Corollary 4.6, if in the previous theorem the set-valued map K is constantly equal to a closed convex proper cone Q, we obtain a particular case of Theorem 3.4 of [5].*

5 Optimality conditions for robust efficient points

In this section we get some optimality conditions for the concepts introduced in Definition 2.4 in terms of Fréchet and Mordukhovich coderivatives of the set-valued maps F and K, respectively. In fact, we deal only with the harder case which is the one of local robust efficiency (Definition 2.4 (i)). Of course, the results we get hold as well in the case of local robust weak efficiency (Definition 2.4 (ii)), when this notion can be defined.

Before we do this we define a notion of openness for a sum of set-valued maps which is weaker than the classical one. More precisely, we say that $F + K$ is weakly open at $(\overline{x}, \overline{y})$ if for every neighborhood U of $\overline{x}$, there exists a neighborhood V of $\overline{y}$ such that $V \subset F(U) + K(U)$. As in general $(F + K)(U) \subset F(U) + K(U)$, if $F + K$ is open in the standard sense at $(\overline{x}, \overline{y})$, then $F + K$ is also weakly open at the same point. However, the reverse implication is not true. In this sense, we give the following example.

Example 5.1 Let $F,\ K : \mathbb{R} \rightrightarrows \mathbb{R}$,

$$F(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}, \\ \emptyset, & \text{if } x \notin \mathbb{Q}, \end{cases}$$

and

$$K(x) = x,\ \forall x \in \mathbb{R},$$

where $\mathbb{Q}$ denotes the set of rational numbers. It is easy to see that $F + K$ is not open in the standard sense at $(0, 0)$, but $F + K$ is weakly open at $(0, 0)$.

The first result of this section is one of weak openness for $F + K$. The method of proof we employ here is well known and can be traced back to [18]. However, due to the specific differences we meet in the case we study, we fill all the technical details of this proof.

Theorem 5.2 *Let X, Y be Asplund spaces, $F,\ K : X \rightrightarrows Y$ set-valued maps, $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ and $(\overline{x}, 0) \in \operatorname{Gr} K$. Assume that the following assumptions are satisfied:*

(i) $\operatorname{Gr} F$ and $\operatorname{Gr} K$ are closed around $(\overline{x}, \overline{y})$ and $(\overline{x}, 0)$, respectively;

(ii) there exist $c > 0$, $r_1 > 0$, $r_2 > 0$, $s_1 > 0$, $s_2 > 0$ such that for every $(x_1, y_1) \in \operatorname{Gr} F \cap (B(\overline{x}, r_1) \times B(\overline{y}, s_1))$, $(x_2, y_2) \in \operatorname{Gr} K \cap (B(\overline{x}, r_2) \times B(0, s_2))$, $y^* \in S_{Y^*}$, z_1^*, $z_2^* \in cB_{Y^*}$, $x_1^* \in \widehat{D}^* F(x_1, y_1)(y^* + z_1^*)$, $x_2^* \in \widehat{D}^* K(x_2, y_2)(y^* + z_2^*)$ we have

$$c \left\| 2y^* + z_1^* + z_2^* \right\| \leq \left\| x_1^* + x_2^* \right\|.$$

Then for every $a \in (0, c)$, there exists $\varepsilon > 0$ such that, for every $\rho \in (0, \varepsilon]$

$$B(\overline{y}, \rho a) \subset F(B(\overline{x}, \rho)) + K(B(\overline{x}, \rho)),$$

and, consequently, $F + K$ is weakly open at $(\overline{x}, \overline{y})$.

Proof. Fix $a \in (0, c)$ and choose $b \in (0, 1)$ such that $\frac{a}{a+1} < b < \frac{c}{c+1}$. There exists $\varepsilon > 0$ such that the following are true:

$$\begin{aligned}
&b^{-1} a\varepsilon < \min\{r_1, r_2, s_1, s_2\}; \\
&\frac{a}{a+1} < b + \varepsilon < \frac{c}{c+1}; \\
&\operatorname{Gr} F \cap \operatorname{cl} W \text{ is closed, where } W = B\left(\overline{x}, b^{-1} a\varepsilon\right) \times B\left(\overline{y}, b^{-1} a\varepsilon\right); \\
&\operatorname{Gr} K \cap \operatorname{cl} V \text{ is closed, where } V = B\left(\overline{x}, b^{-1} a\varepsilon\right) \times B\left(0, b^{-1} a\varepsilon\right).
\end{aligned}$$

Fix $\rho \in (0, \varepsilon]$ and take $v \in B(\overline{y}, \rho a)$. We endow the space $X \times Y \times X \times Y$ with the sum norm and define the function $f : [\operatorname{Gr} F \cap \operatorname{cl} W] \times [\operatorname{Gr} K \cap \operatorname{cl} V] \to \mathbb{R}$, $f(x, y, x_0, y_0) := \|y + y_0 - v\|$. As the set $[\operatorname{Gr} F \cap \operatorname{cl} W] \times [\operatorname{Gr} K \cap \operatorname{cl} V]$ is closed, we can apply the Ekeland Variational Principle for f and $(\overline{x}, \overline{y}, \overline{x}, 0) \in \operatorname{dom} f$. So, there exists $(x_b, y_b, x_{0b}, y_{0b}) \in [\operatorname{Gr} F \cap \operatorname{cl} W] \times [\operatorname{Gr} K \cap \operatorname{cl} V]$ such that

$$\|y_b + y_{0b} - v\| \leq \|\overline{y} - v\| - b\left(\|\overline{x} - x_b\| + \|\overline{y} - y_b\| + \|\overline{x} - x_{0b}\| + \|y_{0b}\|\right) \tag{5.1}$$

and

$$\begin{aligned}
&\|y_b + y_{0b} - v\| \leq \|y + y_0 - v\| + b\left(\|x - x_b\| + \|y - y_b\| + \|x_0 - x_{0b}\| + \|y_0 - y_{0b}\|\right), \\
&\forall (x, y, x_0, y_0) \in [\operatorname{Gr} F \cap \operatorname{cl} W] \times [\operatorname{Gr} K \cap \operatorname{cl} V].
\end{aligned} \tag{5.2}$$

From (5.1) we obtain that

$$\|\overline{x}-x_b\|+\|\overline{y}-y_b\|+\|\overline{x}-x_{0b}\|+\|y_{0b}\|\le b^{-1}\|\overline{y}-v\|<b^{-1}\rho a\le b^{-1}\varepsilon a,$$

whence $(x_b, y_b, x_{0b}, y_{0b}) \in W \times V$. Again from (5.1), if $v = y_b + y_{0b} \in B(\overline{y}, a\rho)$ we obtain that

$$\begin{aligned} b\|\overline{x}-x_b\| &\le \|\overline{y}-v\|-b\left(\|\overline{y}-y_b\|+\|\overline{x}-x_{0b}\|+\|y_{0b}\|\right)\\ &\le \|\overline{y}-v\|-b\|\overline{y}-y_b-y_{0b}\|=(1-b)\|\overline{y}-v\|\\ &\le (1-b)\,a\rho<b\rho \end{aligned}$$

and analogously

$$b\|\overline{x}-x_{0b}\|<b\rho,$$

therefore $x_b,\ x_{0b} \in B(\overline{x},\rho)$. Thus, $v = y_b + y_{0b} \in F(x_b) + K(x_{0b}) \subset F(B(\overline{x},\rho)) + K(B(\overline{x},\rho))$. So, if we prove that v is the only possible solution, we get the conclusion. For this, we suppose that $v \neq y_b + y_{0b}$ and we define the function $h : X \times Y \times X \times Y \to \mathbb{R}$, $h(x,y,x_0,y_0) := \|y+y_0-v\|+b\left(\|x-x_b\|+\|y-y_b\|+\|x_0-x_{0b}\|+\|y_0-y_{0b}\|\right)$. From relation (5.2) we obtain that $(x_b, y_b, x_{0b}, y_{0b})$ is a minimum point for h on $[\operatorname{Gr} F \cap \operatorname{cl} W] \times [\operatorname{Gr} K \cap \operatorname{cl} V]$. Then $(x_b, y_b, x_{0b}, y_{0b})$ is a global minimum point for $h+\delta_{[\operatorname{Gr} F\cap \operatorname{cl} W]\times[\operatorname{Gr} K\cap \operatorname{cl} V]}$. By means of the generalized Fermat rule, we have that

$$(0,0,0,0)\in\widehat{\partial}\left(h(\cdot,\cdot,\cdot,\cdot)+\delta_{[\operatorname{Gr} F\cap \operatorname{cl} W]\times[\operatorname{Gr} K\cap \operatorname{cl} V]}(\cdot,\cdot,\cdot,\cdot)\right)(x_b,y_b,x_{0b},y_{0b}).$$

Since $(x_b, y_b, x_{0b}, y_{0b}) \in W \times V$, we can choose $\gamma \in (0,\rho)$ such that

$$D(x_b,\gamma)\times D(y_b,\gamma)\times D(x_{0b},\gamma)\times D(y_{0b},\gamma)\subset W\times V$$

and

$$v\notin D(y_b+y_{0b},2\gamma).$$

Taking into account that h is Lipschitz and $\delta_{[\operatorname{Gr} F\cap \operatorname{cl} W]\times[\operatorname{Gr} K\cap \operatorname{cl} V]}$ is lower semicontinuous, we can apply the fuzzy calculus rule for the Fréchet subdifferential (Theorem 3.2). Thus, it follows that there exist

$$\begin{aligned} &\left(x_\gamma^1,y_\gamma^1,x_{0\gamma}^1,y_{0\gamma}^1\right)\in D(x_b,\gamma)\times D(y_b,\gamma)\times D(x_{0b},\gamma)\times D(y_{0b},\gamma),\\ &\left(x_\gamma^2,y_\gamma^2,x_{0\gamma}^2,y_{0\gamma}^2\right)\in[\operatorname{Gr} F\cap(D(x_b,\gamma)\times D(y_b,\gamma))]\times[\operatorname{Gr} K\cap(D(x_{0b},\gamma)\times D(y_{0b},\gamma))] \end{aligned}$$

such that

$$\begin{aligned} (0,0,0,0)\in{}&\widehat{\partial}h\left(x_\gamma^1,y_\gamma^1,x_{0\gamma}^1,y_{0\gamma}^1\right)+\widehat{\partial}\delta_{[\operatorname{Gr} F\cap W]\times[\operatorname{Gr} K\cap V]}\left(x_\gamma^2,y_\gamma^2,x_{0\gamma}^2,y_{0\gamma}^2\right)\\ &+\gamma\left(D_{X^*}\times D_{Y^*}\times D_{X^*}\times D_{Y^*}\right). \end{aligned} \tag{5.3}$$

Since $\left(x_\gamma^2,y_\gamma^2,x_{0\gamma}^2,y_{0\gamma}^2\right)\in[\operatorname{Gr} F\cap W]\times[\operatorname{Gr} K\cap V]$ we have that

$$\widehat{\partial}\delta_{[\operatorname{Gr} F\cap W]\times[\operatorname{Gr} K\cap V]}\left(x_\gamma^2,y_\gamma^2,x_{0\gamma}^2,y_{0\gamma}^2\right)=\widehat{N}\left(\operatorname{Gr} F,\left(x_\gamma^2,y_\gamma^2\right)\right)\times\widehat{N}\left(\operatorname{Gr} K,\left(x_{0\gamma}^2,y_{0\gamma}^2\right)\right).$$

As h is the sum of five convex functions, the Fréchet subdifferential $\widehat{\partial}h$ coincides with the sum of the Fenchel subdifferentials.

Now, as in the proof of Theorem 4.9, using the same operator $A : Y \times Y \to Y$ defined by $A(y,z) := y+z$, in composition with the convex function $y \mapsto \|y-v\|$ we have that

$$\partial \|\cdot + \cdot - v\| \left(y_\gamma^1, y_{0\gamma}^1\right) = A^* \left(\partial \|\cdot - v\|\right) \left(y_\gamma^1 + y_{0\gamma}^1\right),$$

where $A^* : Y^* \to Y^* \times Y^*$ denotes the adjoint of A. Remarking also that $v \neq y_\gamma^1 + y_{0\gamma}^1 \in D(y_b + y_{0b}, 2\gamma)$ and as $A^*(y^*) = (y^*, y^*)$ for every $y^* \in Y^*$ we obtain that

$$\partial \|\cdot + \cdot - v\| \left(y_\gamma^1, y_{0\gamma}^1\right) = \left\{(y^*, y^*) \mid y^* \in S_{Y^*}, y^*\left(y_\gamma^1 + y_{0\gamma}^1 - v\right) = \left\|y_\gamma^1 + y_{0\gamma}^1 - v\right\|\right\}.$$

Consequently, using (5.3) we have that

$$\begin{aligned}
(0,0,0,0) \in{} & \{(0, y^*, 0, y^*) \mid y^* \in S_{Y^*}\} + b\left(D_{X^*} \times \{0\} \times \{0\} \times \{0\}\right) + b\left(\{0\} \times D_{Y^*} \times \{0\} \times \{0\}\right) \\
& + b\left(\{0\} \times \{0\} \times D_{X^*} \times \{0\}\right) + b\left(\{0\} \times \{0\} \times \{0\} \times D_{Y^*}\right) \\
& + \widehat{N}\left(\operatorname{Gr} F, \left(x_\gamma^2, y_\gamma^2\right)\right) \times \widehat{N}\left(\operatorname{Gr} K, \left(x_{0\gamma}^2, y_{0\gamma}^2\right)\right) + \gamma\left(D_{X^*} \times D_{Y^*} \times D_{X^*} \times D_{Y^*}\right) \\
={} & (b+\gamma)\left(D_{X^*} \times D_{Y^*} \times D_{X^*} \times D_{Y^*}\right) + \{(0, y^*, 0, y^*) \mid y^* \in S_{Y^*}\} \\
& + \widehat{N}\left(\operatorname{Gr} F, \left(x_\gamma^2, y_\gamma^2\right)\right) \times \widehat{N}\left(\operatorname{Gr} K, \left(x_{0\gamma}^2, y_{0\gamma}^2\right)\right).
\end{aligned}$$

It follows that there exist $y^* \in S_{Y^*}$, $(\widetilde{x}_1^*, \widetilde{y}_1^*, \widetilde{x}_2^*, \widetilde{y}_2^*) \in D_{X^*} \times D_{Y^*} \times D_{X^*} \times D_{Y^*}$ such that

$$\begin{aligned}
\left(-(b+\rho)\widetilde{x}_1^*, -y^* - (b+\rho)\widetilde{y}_1^*\right) &\in \widehat{N}\left(\operatorname{Gr} F, \left(x_\gamma^2, y_\gamma^2\right)\right), \\
\left(-(b+\rho)\widetilde{x}_2^*, -y^* - (b+\rho)\widetilde{y}_2^*\right) &\in \widehat{N}\left(\operatorname{Gr} K, \left(x_{0\gamma}^2, y_{0\gamma}^2\right)\right)
\end{aligned}$$

i.e.,

$$\begin{aligned}
-(b+\rho)\widetilde{x}_1^* &\in \widehat{D}^* F\left(x_\gamma^2, y_\gamma^2\right)\left(y^* + (b+\rho)\widetilde{y}_1^*\right), \\
-(b+\rho)\widetilde{x}_2^* &\in \widehat{D}^* K\left(x_{0\gamma}^2, y_{0\gamma}^2\right)\left(y^* + (b+\rho)\widetilde{y}_2^*\right).
\end{aligned}$$

Using the hypothesis (ii) we obtain

$$c\left\|2y^* + (b+\rho)\left(\widetilde{y}_1^* + \widetilde{y}_2^*\right)\right\| \leq (b+\rho)\left\|\widetilde{x}_1^* + \widetilde{x}_2^*\right\| \leq 2(b+\rho).$$

But

$$c\left\|2y^* + (b+\rho)\left(\widetilde{y}_1^* + \widetilde{y}_2^*\right)\right\| \geq c\left(\|2y^*\| - (b+\rho)\left\|\widetilde{y}_1^* + \widetilde{y}_2^*\right\|\right) \geq 2c\left(1 - (b+\rho)\right),$$

so we have

$$c\left(1 - (b+\rho)\right) \leq b + \rho.$$

This is in contradiction to

$$b + \rho \leq b + \varepsilon < \frac{c}{c+1}.$$

The proof is complete. □

Remark 5.3 *Observe that in the previous theorem we do not require the property of alliedness like in [6, Theorem 4.6] and [17, Theorem 4.2], but our conclusion is weaker.*

In the following lemma, we prove the incompatibility between weakly openness and the optimality notion from Definition 2.4. In general, it is known from the scalar case that a mapping cannot be open at an extremum point. This remark was extended and used in generalized settings in several recent works: see, e.g., [4], [13] and the references therein.

In our specific situation, this can be written as in the following lemma.

Lemma 5.4 *Suppose that there exists a neighborhood U of $\overline{x}$ such that $\bigcap_{x\in U} K(x) \neq \{0\}$. If $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ is a local robust efficient point for F with respect to K, then there exists a neighborhood W of $\overline{x}$ such that for every neighborhood V of $\overline{y}$*

$$V \not\subset F(W) + K(W),$$

i.e., $F+K$ is not weakly open at $(\overline{x},\overline{y})$.

Proof. Using Lemma 2.7, we obtain that there is a neighborhood U_1 of $\overline{x}$ such that, for every x, $z \in U_1$,

$$(F(x) + K(z) - \overline{y}) \cap (-K(z)) \subset \{0\}. \tag{5.4}$$

Take $W = U \cap U_1$. If the conclusion does not hold, then there exists V, a neighborhood of $\overline{y}$, such that $V \subset F(W) + K(W)$. Let $y \in V$. So, there exist z_1, $z_2 \in W$ such that $y \in F(z_1) + K(z_2)$. From (5.4) it follows that $y - \overline{y} \notin -K(z_2)$ or $y - \overline{y} = 0$ i.e.,

$$y - \overline{y} \in \{0\} \cup (Y \setminus -K(z_2)) \subset \{0\} \cup \left(Y \setminus \bigcap_{x\in U} - K(x)\right).$$

But y was chosen arbitrarily in V, so $V - \overline{y} \subset \{0\} \cup \left(Y \setminus \bigcap_{x\in U} - K(x)\right)$. It is easy to see that $V - \overline{y}$ is an absorbing set, i.e., for every $v \in Y$, there is $\gamma > 0$ such that for every $\lambda \in [-\gamma, \gamma]$ we have $\lambda v \in V - \overline{y}$. So, as the set $\{0\} \cup \left(Y \setminus \bigcap_{x\in U} - K(x)\right)$ is a cone, we deduce that $Y = \{0\} \cup \left(Y \setminus \bigcap_{x\in U} - K(x)\right)$, which contradicts the fact that $\bigcap_{x\in U} K(x) \neq \{0\}$. Thus we have the conclusion.
□

Putting together the last lemma and Theorem 5.2, we obtain the following result.

Theorem 5.5 *Let X, Y be Asplund spaces, F, $K : X \rightrightarrows Y$ set-valued maps with $\operatorname{Gr} F$ and $\operatorname{Gr} K$ closed around $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ and $(\overline{x}, 0) \in \operatorname{Gr} K$, respectively. Suppose that there exists a neighborhood U of $\overline{x}$ such that $\overline{K} \neq \{0\}$, where $\overline{K} := \bigcap_{x\in U} K(x)$. If $(\overline{x},\overline{y})$ is a local robust efficient point for F with respect to K, then for every $\varepsilon > 0$, there exist $(x_\varepsilon, y_\varepsilon) \in \operatorname{Gr} F \cap (B(\overline{x},\varepsilon) \times B(\overline{y},\varepsilon))$, $(\widetilde{x}_\varepsilon, \widetilde{y}_\varepsilon) \in \operatorname{Gr} K \cap (B(\overline{x},\varepsilon) \times B(0,\varepsilon))$, $y_\varepsilon^* \in S_{Y^*}$, z_ε^*, $\widetilde{z}_\varepsilon^* \in \varepsilon B_{Y^*}$ such that*

$$0 \in \widehat{D}^* F(x_\varepsilon, y_\varepsilon)(y_\varepsilon^* + z_\varepsilon^*) + \widehat{D}^* K(\widetilde{x}_\varepsilon, \widetilde{y}_\varepsilon)(y_\varepsilon^* + \widetilde{z}_\varepsilon^*) + \varepsilon B_{X^*}.$$

Proof. Using Lemma 5.4 and taking into account the hypotheses, it follows that the second assumption of Theorem 5.2 is not satisfied. Thus, for every $n \in \mathbb{N} \setminus \{0\}$, there exist $(x_n, y_n) \in \operatorname{Gr} F \cap \left(B\left(\overline{x}, \frac{1}{4n}\right) \times B\left(\overline{y}, \frac{1}{4n}\right)\right)$, $(\widetilde{x}_n, \widetilde{y}_n) \in \operatorname{Gr} K \cap \left(B\left(\overline{x}, \frac{1}{4n}\right) \times B\left(0, \frac{1}{4n}\right)\right)$, $(y_n^*) \subset S_{Y^*}$, z_n^*, $\widetilde{z}_n^* \in \frac{1}{4n} B_{Y^*}$, $x_n^* \in \widehat{D}^* F(x_n, y_n)(y_n^* + z_n^*)$, $\widetilde{x}_n^* \in \widehat{D}^* K(\widetilde{x}_n, \widetilde{y}_n)(y_n^* + \widetilde{z}_n^*)$ such that

$$\|x_n^* + \widetilde{x}_n^*\| < \frac{1}{4n} \|2y_n^* + z_n^* + \widetilde{z}_n^*\| < \frac{4n+1}{8n^2} < \frac{1}{n}.$$

Since, for every $\varepsilon > 0$ we can find $n \in \mathbb{N} \setminus \{0\}$ with $\frac{1}{n} < \varepsilon$, we obtain the conclusion. □

To obtain optimality conditions in terms of Mordukhovich coderivatives, we need a sequentially normally boundedness condition. We say that a set-valued map $H : X \rightrightarrows Y$ is sequentially normally bounded ((SNB), for short) at $(\overline{x}, \overline{y}) \in \operatorname{Gr} H$ if for every sequences $(x_n, y_n) \overset{\operatorname{Gr} H}{\to} (\overline{x}, \overline{y})$, (y_n^*) bounded and $x_n^* \in \widehat{D}^* H(x_n, y_n)(y_n^*)$ imply that (x_n^*) is bounded too. Also, in order to get nontrivial Lagrange multipliers for the objective we need a generalized compactness hypothesis. Let $C \subset Y$ be a closed set around $c \in C$. One says that C is sequentially normally compact ((SNC), for short) at c if for any $c_n \overset{C}{\to} c$, $c_n^* \overset{w^*}{\to} 0$, $c_n^* \in \widehat{N}(C, c_n)$, we have $c_n^* \to 0$. When C is a proper closed convex cone, the previous property at 0 is given as follows:

$$\left[(c_n^*) \subset C^*, c_n^* \overset{w^*}{\to} 0\right] \Rightarrow c_n^* \to 0.$$

Theorem 5.6 *Suppose that the same hypotheses hold as in Theorem 5.5 and the following two assumptions are satisfied:*

(i) $\overline{K}$ is (SNC) at 0;

(ii) K satisfies the condition (SNB) at $(\overline{x}, 0)$ or F satisfies the condition (SNB) at $(\overline{x}, \overline{y})$.

Then there exists $y^* \in \overline{K}^* \setminus \{0\}$ such that

$$0 \in D_N^* F(\overline{x}, \overline{y})(y^*) + D_N^* K(\overline{x}, 0)(y^*). \tag{5.5}$$

Proof. From the above theorem we obtain that for every $n \in \mathbb{N}\setminus\{0\}$, there exist $(x_n, y_n) \overset{\operatorname{Gr} F}{\to} (\overline{x}, \overline{y})$, $(\widetilde{x}_n, \widetilde{y}_n) \overset{\operatorname{Gr} K}{\to} (\overline{x}, 0)$, $(y_n^*) \subset S_{Y^*}$, (z_n^*), $(\widetilde{z}_n^*) \to 0$, $x_n^* \in \widehat{D}^* F(x_n, y_n)(y_n^* + z_n^*)$, $\widetilde{x}_n^* \in \widehat{D}^* K(\widetilde{x}_n, \widetilde{y}_n)(y_n^* + \widetilde{z}_n^*)$ such that

$$x_n^* + \widetilde{x}_n^* \to 0. \tag{5.6}$$

As $(y_n^*) \subset S_{Y^*}$, we obtain that there exists y^* such that $y_n^* \overset{w^*}{\to} y^*$ (without relabelling), and from Proposition 3.5 (i) we get that $y_n^* + \widetilde{z}_n^* \in (K(\widetilde{x}_n))^*$. Now, taking into account that $\widetilde{x}_n \to \overline{x}$, for n large enough we have that $\widetilde{x}_n \in U$, so $\overline{K} \subset K(\widetilde{x}_n)$. It follows that $y_n^* + \widetilde{z}_n^* \in \overline{K}^*$, and as $\overline{K}^*$ is weakly-star closed and $y_n^* + \widetilde{z}_n^* \overset{w^*}{\to} y^*$, we get that $y^* \in \overline{K}^*$. Suppose first that K satisfies the condition (SNB) at $(\overline{x}, 0)$, hence $\widetilde{x}_n^*$ is bounded. Whence there exists $\widetilde{x}^* \in X^*$ such that $\widetilde{x}_n^* \overset{w^*}{\to} \widetilde{x}^*$. Thus, from (5.6) we obtain that there exists $x^* \in X^*$ such that $x_n^* \overset{w^*}{\to} x^*$ and $\widetilde{x}^* + x^* = 0$. Using the definition of the Mordukhovich coderivative, it follows that $0 \in D_N^* F(\overline{x}, \overline{y})(y^*) + D_N^* K(\overline{x}, 0)(y^*)$. A similar argument works if F satisfies the condition (SNB) at $(\overline{x}, \overline{y})$. To complete the proof, we only need to prove that $y^* \neq 0$. For this, we suppose that $y^* = 0$. Using the (SNC) property of $\overline{K}$ at 0, we obtain from $y_n^* + \widetilde{z}_n^* \in \overline{K}^*$ and $y_n^* + \widetilde{z}_n^* \overset{w^*}{\to} 0$, that $y_n^* + \widetilde{z}_n^* \to 0$. As $\widetilde{z}_n^* \to 0$, it results that (y_n^*) also converges to 0, which contradicts the inclusion $(y_n^*) \subset S_{Y^*}$. □

Remark 5.7 *(i) The hypothesis (i) for the above theorem is satisfied when $\operatorname{int} \overline{K} \neq \emptyset$ or the dimension of Y is finite (for more details, see [21]).*

(ii) According to [15, Theorem 1.43], if F is Lipschitz-like around $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ or K is Lipschitz-like around $(\overline{x}, 0) \in \operatorname{Gr} K$, then the hypothesis (ii) for the above theorem is satisfied.

6 Final comments

In the main results of Sections 4 and 5 we have deduced optimality conditions in terms of Fréchet and Mordukhovich coderivatives of the set-valued maps F and K, respectively. This unites the study we made on both efficiency notions defined in Section 2. So, as we have seen, to obtain optimality conditions in terms of Fréchet coderivative, in both sections we defined a function for which we found a minimum point in a certain domain. Then, in both cases, we used the infinite penalization technique, the generalized Fermat rule and some calculus for the Fréchet subdifferential. For getting optimality conditions in terms of Mordukhovich coderivative, we started from the optimality conditions in terms of Fréchet coderivative and we used various other conditions for each individual case, like the alliedness property in Section 4, or a sequentially normally boundedness condition in Section 5. Regarding the final conclusions of Sections 4 and 5, they are comparable. For example, the right-hand sides of relations (4.4) and (5.5) look similar and they are also in relation with the conclusion of [6, Theorem 4.10]. However, one can observe different combinations of the technical assumptions made in every particular case. That interplay of the assumptions is, on one hand, interesting, and, on the other hand, marks the specificity of every of the efficiencies we study in this paper.

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