Risk Theory and Actuariat

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Chapter 1. No-Arbitrage and the Fundamental Theorem of Asset Pricing

In this chapter, we develop a mathematical model for financial markets in one period, introduce the key concept of no-arbitrage, and formulate and prove the so-called *Fundamental Theorem of Asset Pricing* on the absence of arbitrage in this setting. We shall assume that we have a frictionless market. This means that there are no transaction costs, i.e., assets can be bought and sold at the same price, and there are no constraints on the number of assets one holds. In particular, one can hold a negative amount of some asset, i.e., assets can be shorted and the price paid/received is linear in the quantity of assets bought/sold. Moreover, we shall assume that asset prices are exogeneously given and not in influenced by the trading activities of other market participants. Thus agents are views as price takers. Our aim is to give a necessary and sufficient condition on the market to be arbitrage free. To this end, we introduce two concepts. The first concept is the notion of *discounting.* Assets are denoted in units of something, e.g. EUR. Notwithstanding, it is clear that prices (and values) are relative. So basic concepts of financial markets (like being arbitrage-free) should not and do not depend on the choice of unit. For this reason, we are free to change the unit, in particular if this makes the mathematics simpler. A good choice is a unit which itself is a traded asset, and the canonical choice is to use a risk-free asset on the market.

Chapter 2. Mean-Variance Portfolio Selection and the CAPM.

In this chapter, we give the answer to the question how to optimally invest in a financial market taking into account the mean and the variance of the return of a portfolio. We then deduce that if all market participants behave optimally in a mean-variance sense, the financial market has a special structure, which is described by the *Capital Asset Pricing Model (CAPM).*

Maximizing the expected return alone is not a good criterion for portfolio choice since it does not control the risk inherent in an investment. Markowitz, in a seminal work in 1952 (for which he was awarded the Nobel Prize in Economics in 1990), proposed to consider the variance of the return as a measure of the risk of a portfolio and introduced what is now known as mean-variance portfolio selection. There are two versions of the mean-variance problem, each of which has a formulation with risk-only and one with general portfolios, i.e., portfolios which also allow an investment in the riskless asset:

- Given an initial wealth $x_0 > 0$ and a minimal desired expected return $\mu_{min} > 0$, minimize the variance of the return $\sigma_{\bar{v}}^2$ among all x_0-feasible portfolios $\bar{v} \in \mathbb{R}^{d+1}$ that satisfy $\mu_{\bar{v}} \geq \mu_{min}$.
- Given an initial wealth $x_0 > 0$ and a minimal desired expected return $\mu_{min} > 0$, minimize the variance of the return σ_v^2 among all risk-only x_0-feasible portfolios $v \in \mathbb{R}^d$ that satisfy $\mu_v \geq \mu_{min}$.
- Given an initial wealth $x_0 > 0$ and a maximal variance of the return $\sigma_{\max}^2 \geq 0$, maximize the expected return $\mu_{\bar{v}}$ among all x_0-feasible portfolios $\bar{v} \in \mathbb{R}^{d+1}$ that satisfy $\sigma_{\bar{v}}^2 \leq \sigma_{\max}^2$.
- Given an initial wealth $x_0 > 0$ and a maximal variance of the return $\sigma_{\max}^2 \geq 0$, maximize the expected return μ_v among all risk-only x_0-feasible portfolios $v \in \mathbb{R}^d$ that satisfy $\sigma_v^2 \leq \sigma_{\max}^2$.

We introduce the key concept of a *risk-only efficient portfolio* and define the *risk-only efficient frontier* and we give its characterization. We also introduce and analyze the *Markowitz tangency portfolio* and discuss about the *Capital Market Line.*

Chapter 3. Utility Theory

In this chapter, we describe preferences of an investor who has to compare random outcomes like the future payoff of a financial asset or the return of a portfolio. To this end, we will follow the axiomatic approach proposed by von Neumann and Morgenstern. From a mathematical perspective, it is easier to describe preferences on the level of probability distributions than on the level of random variables (even though the latter might be more intuitive from an economic perspective).

Mathematically, the definition of a preference order is satisfactory. From a practical perspective, however, it is very unhandy because we need to specify for each pair of lotteries ν_1 and ν_2, whether we weakly prefer ν_1 over ν_2, or ν_2 over ν_1, or both. For this reason, we seek to find another description of preference orders that encodes preferences by a single mathematical object. Von Neumann and Morgenstern showed that many preference orders can be described by specifying a single function. Our goal is to introduce axioms for preference orders that together imply a von Neumann-Morgenstern representation. Essentially, only two axioms are needed to ensure a von Neumann-Morgenstern representation: *the independence axiom* and *the continuity axiom.* After this, we introduce and discuss the concept of lotteries and the implied risk adversion. For doing this, some convex and concave optimization problems are considered.

Chapter 4. Introduction to Risk Measures

We discuss how to quantify the downside risk of a financial position. To this end, we follow the axiomatic approach initiated by Artzner, Delbaen, Ebner and Heath. Convexity of a risk measure formalizes the idea that diversification should reduce the risk. This is best seen in the equivalent formulation as subadditivity (assuming also positive homogeneity): If a large company has different product lines or desks, the total risk of the aggregate position is bounded by the sum of the individual risks related to each product line or desk . Apart from being a reasonable requirement from an economic perspective, this property is quite useful from a management perspective. Positive homogeneity of a risk measure means that risk grows in a linear way.

We discuss the two most important *risk measures* used in practice. The first example is *value at risk* which was introduced in 1993 and widely propagated by the "RiskMetrics" of JP Morgan launched in 1994. However, Value at Risk fails to be convex, i.e., it may punish diversification instead of encouraging it. For this reason, one might discuss about a second example, more conservative, i.e., larger risk measure than Value at Risk, which also takes into account the size of the loss given default. *Expected Shortfall* is a more coherent risk measure, i.e., it is in particular a convex risk measure. One can even show that it is optimal in the sense that it is the smallest law-invariant convex risk measure that is more conservative than Value at Risk.

Chapter 5. Pricing and Hedging in Finite Discrete Time

The goal of this chapter is to shortly describe pricing and hedging of derivative contracts like call or put options of financial markets in finite discrete time. One of the key objects used in the study of mathematical finance concepts are the martingales. In order to define and study them, we first need to introduce and analyze the conditional expectation with respect to a $\sigma-$algebra. Some basic examples prepare the more advanced study of pricing financial derivatives in discrete and continuous financial markets. For doing this, stochastic analysis tools are mandatory.